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Picture Hesitant Fuzzy Generalized Dice Similarity Measures and Their Application in Pattern Recognitions

Z. Ali[1], T. Mahmood [2]

[1,2] *Department of Mathematics and Statistic, International Islamic University Islamabad, Pakistan;*

[1] zeeshan.phdma102@iiu.edu.pk.

Abstract: The idea of picture hesitant fuzzy set (PHFS) is the combination of picture fuzzy set (PFS) and hesitant fuzzy set (HFS), which is characterized by three functions representing the degree of membership, the degree of abstinence and the degree of non-membership are the form of subsets of unit interval. The idea of PHFS is an important tool to deal with abstinence and inconsistence information in real life problems. Motivations of this manuscript, we presented some new dice similarity measures (DSMs) for PHFSs, generalization dice similarity measures (GDSMs) for PHFSs and specify that the DSM and asymmetric measures (projection measures) are the particular case of the GDSM in some parameter values. Next, we investigated the GDSM based patterns recognition models with picture hesitant fuzzy information (PHFI). Then, with the help of GDSM for PHFSs to developed the application in building material recognition. Moreover, we evaluate the numerical example to show the effectiveness and flexibility of the similarity measures (SMs) in the environment of building metrical recognitions. In last, we will discuss the comparison between proposed methods with existing methods with the help of some examples, which shown the flexibility and effectiveness of the proposed methods.

Keywords: Dice Similarity Measures; Generalized Dice Similarity Measures; Measures, Picture Fuzzy sets; Picture Hesitant Fuzzy Sets.

I. INTRODUCTION

In 1965, the idea of fuzzy set (FS) proposed by Zadeh [1] is an helpful tool to deal with difficult information in real words. FS is characterized by only membership function, which is bounded to [0,1]. A-intuitionistic fuzzy set (A-IFS) proposed by Atanassove [2], which is a generalization of FS is able to deal with uncertain and critical information. A-IFS described the opinion of the human being like true or false, denoted the membership and non-membership function, whose sum is bounded to unit interval [0,1]. A-IFS extensively used in many fields [3, 4]. In 2002, Xu [5] proposed the intuitionistic fuzzy aggregation operators while Garg [6] presented generalized intuitionistic fuzzy soft power aggregation operator based on t-norm and their application in multicriteria decision making. In 2006, Xu and Yager [7], proposed some geometric aggregation operators based on intuitionistic fuzzy sets and Wang and Lui [8] introduced the notion of intuitionistic fuzzy information aggregation using Einstein operators. Further, Ejegwa et al. [9] initiated application of intuitionistic fuzzy set in electrol system. While, Kang [10] introduced the notion of New hesitation-based distance and similarity measures on intuitionistic fuzzy setsand their applications. Recently, Garg and Singh [11] proposed a novel triangular interval type-2 intuitionistic fuzzy sets and their aggregation operators.

A-IFS has been successfully applied in different fields, but there are some problems in real life, the A-IFS cannot able to handle such kind of information. For example: in the duration of casting of voting, the human being have different opinion like: agree, non-appearance, disagree and refusal. For handling this types of issue Cuong [12] initiated the notion of picture fuzzy set (PFS), which is an extension of A-IFS. The notion PFS $I = \{(\mathfrak{X}, \mathfrak{M}_I(\mathfrak{X}), \mathfrak{A}_I(\mathfrak{X}), \mathfrak{N}_I(\mathfrak{X})) : \mathfrak{X} \in X\}$ characterized by membership, abstinence and non-membership function belonging to [0,1] with a condition $0 \leq \mathfrak{M}_I(\mathfrak{X}) + \mathfrak{A}_I(\mathfrak{X}) + \mathfrak{N}_I(\mathfrak{X}) \leq 1$. PFS is a grateful tool to deal with unpredictable and undetermined information in the environment of A-IFS and FS. When the decision maker is provides (0.6,0.1,0.2) represented the picture fuzzy number (PFN), such type of problem cannot handle by FS and A-IFS. Moreover, many researchers have developed to solve the problem under the environment of PFSs. In 2015, Singh [13] investigated the concept of correlation coefficients for picture fuzzy set and Son [14] introduced a generalized picture distance measure and application to picture fuzzy clustering. Moreover, Wei [15] proposed a new direction in picture fuzzy set like picture fuzzy cross-entropy for multi-attribute decision making problems. Currently, Wei and Garg [16,17] proposed some picture fuzzy aggregation operators and their applications to multicriteria decision making while Wei [18] initiated picture fuzzy heronian mean aggregation operators in multi attribute decision making. For more theoretical and experimental work in PFSs, we suggested the following Ref. [19, 20].

Hesitant fuzzy set (HFS) proposed by Torra [21] as a generalization of FS. Basically, HFS characterized by membership function is the form of subset of unit interval [0,1]. The concept of HFS is a new direction for researchers to deal with unknown and unresolved information in real decision problems. In 2017, Wei [22] developed the notion of hesitant bipolar aggregation operators in multi attribute decision making. Moreover, Faizi [23] proposed the decision-making with uncertainty using hesitant fuzzy sets while Zhou [24] presented multi-criteria decision-making approaches based on distance measures for linguistic hesitant fuzzy sets. For theoretical and practical work on HFSs see Ref. [25]. Beg and Rashid [26] proposed the notion of Intuitionistic hesitant fuzzy set (IHFS), which are the combination of A-IFSs and HFSs. IHFS is able to solve these types of issues like A-IFSs, FS and HFSs, but the FSs, A-IFSs and HFSs are not skillful concepts to handle PHFSs types of information. Picture hesitant fuzzy set (PHFS) proposed by Ullah et al. [27] in 2018, which is the collection of PFSs and HFSs. The notion of PHFSs is more generalized then IHFSs.

Picture hesitant fuzzy set is the mixture of picture fuzzy sets and hesitant fuzzy set to cope with uncertain and complicated information in realistic decision issues. There was many situations, the notions of IFSs, PFSs, HFSs, and their combinations are cannot working effectively. For instance, when a decision maker provides the grade of truth in the form of $\{0.1,0.2,0.3\}$, the grade of abstinence in the form of $\{0.2,0.3,0.4\}$, and the grade of falsity in the form of $\{0.01,0.02,0.03\}$, then the intuitionistic fuzzy set, intuitionistic hesitant fuzzy set, picture fuzzy set, and etc. cannot cope with it. For coping with such types of issues, the theory of picture hesitant fuzzy set is very beneficial to cope with it. Because the condition of picture hesitant fuzzy set is that the sum of the maximum of the truth grade, abstinence grade, and falsity grade is cannot exceeded form unit interval. The concept of similarity is a basic concept in human cognition. Similarity plays an essential role in taxonomy, recognition, case-based reasoning and many other fields. There are many aspects of the concept of similarity that have eluded formalization. According to [hesitant fuzzy set] formulation of a valid, general purpose definition of similarity is a challenging problem. There does not exist a valid, general-purpose definition of similarity. There does exist many special purpose definitions which have been employed with success in cluster analysis, search, classification, recognition and diagnostics. There are several similarity measures that are proposed and used for varied purposes. The similarity measures are classified into three categories: 1) Metric based measures, 2) Set-theoretic based measures and 3) Implicators based measures. While dealing with distance based similarity measures, examples have been constructed for perceptual similarity where, every distance axiom is clearly violated by dissimilarity measures particularly the triangle inequality and consequently the corresponding similarity measure disobeys transitivity. This model postulates that the perceptual distance satisfies the metric axioms, the empirical validity of which has been experimentally challenged by several researchers, particularly the triangle inequality. Similarly in case of set theoretic similarity measures, it is observed that crisp transitivity is a much stronger condition to be put upon similarity measure. Set theoretic similarity measures are further subdivided in three groups (a) measures based on crisp logic (b) measures based on fuzzy logic (c) measures based on hesitant fuzzy sets. The advantages of PHFSs are discussed in the following way:

1. PHFSs are converted to PFSs proposed by Coung [12], when the membership, the abstinence and the non-membership grades will be consider to singleton sets.
2. PHFSs are converted to IHFSs proposed by Bed and Rashid [26], when the abstinence grades will be consider to zero.
3. PHFSs are converted to IFSs proposed by Atanassove [2], when the membership and the

non-membership grades will be consider to singleton sets and the abstinence grades will be consider to zero.

4. PHFSs are converted to HFSs proposed by Torra [21], when the non-membership and the abstinence grades will be consider to zero.
5. PHFSs are converted to FSs proposed by Zadeh [1], when the membership grades will be consider to singleton sets and the abstinence, non-membership grades will be consider to zero.

The remainder of this paper, introduction is the first of this manuscript. In the second part, we proposed some new DSMs and WDSM for PHFS. In the third part of this manuscript, we modified the DSM and WDSM into GDSM and WGDSM for PHFS, we also discussed the special cases of our proposed work whose name is projection/asymmetric measures for PHFS. In the fourth part, with the help of WGDSM for PHFS we solve a problem of building material recognition to demonstrate the effectiveness and reliability of our proposed work. In the fifth part, we described the comparative study of PHFS with pre-existing work. In the sixth part, we discussed the advantages of PHFSs with examples. In the last part of this manuscript, we derived a conclusion with the help of some remarks.

II. PRELIMINARIES

In this section, we discussed the basic concepts of intuitionistic fuzzy sets, picture fuzzy sets, hesitant fuzzy set and picture hesitant fuzzy set. Throughout this manuscript X denote the universal set and $\mathfrak{M}$, $\mathfrak{A}$ and $\mathfrak{N}$ are representing the membership, abstinence and non-membership grade. Further $PFS(X)$ denotes the set of PFS and $P\boldsymbol{\mathcal{H}}FS(X)$ denote the set of all PHFSs. Throughout this manuscript $\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{H}}' \in P\boldsymbol{\mathcal{H}}FS(X)$. Moreover, $w = (w_1, w_2, \dots, w_M)^T$ is called weight vector, where $w_i \in [0,1]$, $\sum_{i=1}^{M} w_i = 1$.

Definition 1 [2]: An IFS I is of the form $I = \{(\mathfrak{X}, \mathfrak{M}_I(\mathfrak{X}), \mathfrak{N}_I(\mathfrak{X})) : \mathfrak{X} \in X\}$, which is hold the following condition: $0 \le \mathfrak{M}_I(\mathfrak{X}) + \mathfrak{N}_I(\mathfrak{X}) \le 1$ for all $\mathfrak{X} \in X$. Moreover for all $\mathfrak{X} \in X$, $d_I(\mathfrak{X}) = 1 - (\mathfrak{M}_I(\mathfrak{X}) + \mathfrak{N}_I(\mathfrak{X}))$ is represented the hesitancy degree of $\mathfrak{X}$ in I. The intuitionistic fuzzy number (IFN) is denoted by $(\mathfrak{M}, \mathfrak{N})$.

Definition 2 [12]: A PFS I is of the form $I = \{(\mathfrak{X}, \mathfrak{M}_I(\mathfrak{X}), \mathfrak{A}_I(\mathfrak{X}), \mathfrak{N}_I(\mathfrak{X})) : \mathfrak{X} \in X\}$, which is hold the following condition: $0 \le \mathfrak{M}_I(\mathfrak{X}) + \mathfrak{A}_I(\mathfrak{X}) + \mathfrak{N}_I(\mathfrak{X}) \le 1$ for all $\mathfrak{X} \in X$. Moreover for all $\mathfrak{X} \in X$, $r_I(\mathfrak{X}) = 1 - (\mathfrak{M}_I(\mathfrak{X}) + \mathfrak{A}_I(\mathfrak{X}) + \mathfrak{N}_I(\mathfrak{X}))$ is represented the refusal degree of $\mathfrak{X}$ in I. The picture fuzzy number (PFN) is denoted by $(\mathfrak{M}, \mathfrak{A}, \mathfrak{N})$.

Definition 3 [21]: A HFS is a mapping S that gives us few elements of [0,1] against each $\mathfrak{X} \in X$. i.e $\boldsymbol{\mathcal{H}} = \{(\mathfrak{X}, \mathfrak{M}(\mathfrak{X})) : \mathfrak{M}(\mathfrak{X})$ is a finite subset of $[0,1]\ \forall\ \mathfrak{X} \in X\}$. Moreover, $\mathfrak{M}$ is called hesitant fuzzy number (HFN).

Definition 4 [27]: A PHFS $\boldsymbol{\mathcal{H}}$ is of the form $\boldsymbol{\mathcal{H}} = \{(\mathfrak{X}, \mathfrak{M}_{\boldsymbol{\mathcal{H}}}(\mathfrak{X}), \mathfrak{A}_{\boldsymbol{\mathcal{H}}}(\mathfrak{X}), \mathfrak{N}_{\boldsymbol{\mathcal{H}}}(\mathfrak{X})) : \mathfrak{X} \in X\}$, where $\mathfrak{M}_{\boldsymbol{\mathcal{H}}}(\mathfrak{X})$, $\mathfrak{A}_{\boldsymbol{\mathcal{H}}}(\mathfrak{X})$ and $\mathfrak{N}_{\boldsymbol{\mathcal{H}}}(\mathfrak{X})$ be a hesitant fuzzy numbers (HFNs), which is hold the following condition: $0 \le \mathfrak{MAX}(\mathfrak{M}_{\boldsymbol{\mathcal{H}}}(\mathfrak{X})) + \max(\mathfrak{A}_{\boldsymbol{\mathcal{H}}}(\mathfrak{X})) + \max(\mathfrak{N}_{\boldsymbol{\mathcal{H}}}(\mathfrak{X})) \le 1$ for all $\mathfrak{X} \in X$. Moreover for all $\mathfrak{X} \in X$, $\pi_{\boldsymbol{\mathcal{H}}}(\mathfrak{X}) = 1 - (\mathfrak{MAX}(\mathfrak{M}_{\boldsymbol{\mathcal{H}}}(\mathfrak{X})) + \max(\mathfrak{A}_{\boldsymbol{\mathcal{H}}}(\mathfrak{X})) + \max(\mathfrak{N}_{\boldsymbol{\mathcal{H}}}(\mathfrak{X})))$ is represented the refusal degree of $\mathfrak{X}$ in $\boldsymbol{\mathcal{H}}$.

Definition 5 [28]: Let X and Y be a two vectors, whose length are finite and positive, the DSM is of the form $D(X,Y) = \frac{2X.Y}{\|X\|_2^2 + \|Y\|_2^2} = \frac{2\sum_{j=1}^{l} \mathfrak{X}_j y_j}{\sum_{j=1}^{l} \mathfrak{X}_j^2 + \sum_{j=1}^{l} y_j^2}$, where $X.Y = \sum_{j=1}^{l} \mathfrak{X}_j y_j$ is represented the inner product and $\|X\|_2 = \sqrt{\sum_{j=1}^{l} \mathfrak{X}_j^2}$ and $\|Y\|_2 = \sqrt{\sum_{j=1}^{l} y_j^2}$ is represented the Euclidean or L_2 norms of X and Y.

Definition 6 : A DSM for PHFSs $D^1{}_{P\boldsymbol{\mathcal{H}}F}(\boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{H}}')$ is of the form

$D^1{}_{P\boldsymbol{\mathcal{H}}F}(\boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{H}}')$

$$=\frac{1}{M}\sum_{i=1}^{M}\frac{2\left(\frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l}\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\frac{1}{\dot{\mathfrak{A}}}\sum_{j=1}^{l}\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\frac{1}{\dot{\mathfrak{N}}}\sum_{j=1}^{l}N_{\mathcal{H}}^{j}(\mathfrak{x}_i)N_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)}{\left(\begin{array}{c}\frac{1}{L_{\mathfrak{M}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{N}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\\ \frac{1}{L_{\mathfrak{M}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{N}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2\end{array}\right)} \quad (1)$$

Which satisfy following conditions:

1. $0 \leq D^1{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') \leq 1$
2. $D^1{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') = D^1{}_{P\mathcal{H}F}(\mathcal{H}',\mathcal{H})$
3. $D^1{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') = 1 \Leftrightarrow \mathcal{H} = \mathcal{H}'$

Definition 7 **:** A WDSM for PHFSs $WD^1{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}')$ is of the form

$WD^1{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}')$

$$=\sum_{i=1}^{M}w_i\frac{2\left(\frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l}\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\frac{1}{\dot{\mathfrak{A}}}\sum_{j=1}^{l}\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\frac{1}{\dot{\mathfrak{N}}}\sum_{j=1}^{l}N_{\mathcal{H}}^{j}(\mathfrak{x}_i)N_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)}{\left(\begin{array}{c}\frac{1}{L_{\mathfrak{M}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{N}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\\ \frac{1}{L_{\mathfrak{M}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{N}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2\end{array}\right)} \quad (2)$$

Which satisfy following conditions:

1. $0 \leq WD^1{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') \leq 1$
2. $WD^1{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') = WD^1{}_{P\mathcal{H}F}(\mathcal{H}',\mathcal{H})$
3. $WD^1{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') = 1 \Leftrightarrow \mathcal{H} = \mathcal{H}'$

*Remarks 1***:** If we take $w = (\frac{1}{M},\frac{1}{M},\dots,\frac{1}{M})^T$ then the WDSM for PHFS reduced to DSM for PHFS i.e. $WD^1{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') = D^1{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}')$. Moreover we are proposed the other form of DSM and WDSM for PHFS, if we including the fourth terms like refusal degree.

III. OTHER FORM OF DSM AND WDSM FOR PHFS

*Definition 8***:** A DSM for PHFSs $D^2{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}')$ is of the form

$D^2{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}')$

$$= \frac{1}{M}\sum_{i=1}^{M}\frac{2\begin{pmatrix}\frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l}\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\frac{1}{\dot{\mathfrak{A}}}\sum_{j=1}^{l}\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\\ \frac{1}{\dot{\mathfrak{N}}}\sum_{j=1}^{l}N_{\mathcal{H}}^{j}(\mathfrak{x}_i)N_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\frac{1}{\dot{\pi}}\sum_{j=1}^{l}\pi_{\mathcal{H}}^{j}(\mathfrak{x}_i)\pi_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\end{pmatrix}}{\begin{pmatrix}\frac{1}{L_{\mathfrak{M}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{N}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2\\ +\frac{1}{L_{\pi_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\pi_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{M}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2\\ +\frac{1}{L_{\mathfrak{A}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{N}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\pi_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\pi_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2\end{pmatrix}} \quad (3)$$

Which satisfy the following conditions:

1. $0 \le D^2{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') \le 1$
2. $D^2{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') = D^2{}_{P\mathcal{H}F}(\mathcal{H}',\mathcal{H})$
3. $D^2{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') = 1 \Leftrightarrow \mathcal{H} = \mathcal{H}'$

*Proof***:** Firstly, we prove the conditions (1), we have

$D^2{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') \ge 0$, are obviously holds. Next, to prove that $D^2{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') \le 1$, for this

$$\begin{pmatrix}\frac{1}{L_{\mathfrak{M}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{N}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2\\ +\frac{1}{L_{\pi_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\pi_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{M}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2\\ +\frac{1}{L_{\mathfrak{A}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{N}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\pi_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\pi_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2\end{pmatrix}$$

$$\ge 2\begin{pmatrix}\frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l}\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\frac{1}{\dot{\mathfrak{A}}}\sum_{j=1}^{l}\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\\ \frac{1}{\dot{\mathfrak{N}}}\sum_{j=1}^{l}N_{\mathcal{H}}^{j}(\mathfrak{x}_i)N_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\frac{1}{\dot{\pi}}\sum_{j=1}^{l}\pi_{\mathcal{H}}^{j}(\mathfrak{x}_i)\pi_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\end{pmatrix}$$

To the help $a^2 + b^2 \ge 2ab$, so $0 \le D^2{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') \le 1$ is holds.

Next, we prove the condition (2), which is obviously holds.

In last, we prove the condition (3), we assume that $\mathcal{H} = \mathcal{H}'$. Then to prove that $D^2{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') = 1$.

By hypothesis $\mathcal{H} = \mathcal{H}'$, it mean that every set in $\mathcal{H}$ is equal to the every set in $\mathcal{H}'$, so

$D^2{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}')$

$$= \frac{1}{M}\sum_{i=1}^{M}\frac{2\left(\begin{array}{c}\frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l}\mathfrak{M}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\mathfrak{M}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)+\frac{1}{\dot{\mathfrak{A}}}\sum_{j=1}^{l}\mathfrak{A}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\mathfrak{A}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)+\\ \frac{1}{\dot{\mathfrak{N}}}\sum_{j=1}^{l}N_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)N_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)+\frac{1}{\dot{\pi}}\sum_{j=1}^{l}\pi_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\pi_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\end{array}\right)}{\left(\begin{array}{c}\frac{1}{L_{\mathfrak{M}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{N}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2\\ +\frac{1}{L_{\pi_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\pi_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{M}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2\\ +\frac{1}{L_{\mathfrak{A}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{N}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\pi_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\pi_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2\end{array}\right)}$$

$$= \frac{1}{M}\sum_{i=1}^{M}\frac{2\left(\begin{array}{c}\frac{1}{L_{\mathfrak{M}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2+\\ \frac{1}{L_{\mathfrak{N}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\pi_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\pi_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2\end{array}\right)}{2\left(\begin{array}{c}\frac{1}{L_{\mathfrak{M}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2+\\ \frac{1}{L_{\mathfrak{N}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\pi_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\pi_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2\end{array}\right)} = 1$$

The proof is complete.

*Definition 9***:** A WDSM for PHFSs $WD^2{}_{P\boldsymbol{\mathcal{H}}F}(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{H}}')$ is of the form

$WD^2{}_{P\boldsymbol{\mathcal{H}}F}(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{H}}')$

$$= \sum_{i=1}^{M} w_i\frac{2\left(\begin{array}{c}\frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l}\mathfrak{M}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\mathfrak{M}_{\boldsymbol{\mathcal{H}}'}^{j}(\mathfrak{x}_i)+\frac{1}{\dot{\mathfrak{A}}}\sum_{j=1}^{l}\mathfrak{A}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\mathfrak{A}_{\boldsymbol{\mathcal{H}}'}^{j}(\mathfrak{x}_i)+\\ \frac{1}{\dot{\mathfrak{N}}}\sum_{j=1}^{l}N_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)N_{\boldsymbol{\mathcal{H}}'}^{j}(\mathfrak{x}_i)+\frac{1}{\dot{\pi}}\sum_{j=1}^{l}\pi_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\pi_{\boldsymbol{\mathcal{H}}'}^{j}(\mathfrak{x}_i)\end{array}\right)}{\left(\begin{array}{c}\frac{1}{L_{\mathfrak{M}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{N}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2\\ +\frac{1}{L_{\pi_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\pi_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{M}_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\boldsymbol{\mathcal{H}}'}^{j}(\mathfrak{x}_i)\right)^2\\ +\frac{1}{L_{\mathfrak{A}_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\boldsymbol{\mathcal{H}}'}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{N}_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\boldsymbol{\mathcal{H}}'}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\pi_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\pi_{\boldsymbol{\mathcal{H}}'}^{j}(\mathfrak{x}_i)\right)^2\end{array}\right)} \quad (4)$$

Which satisfy following conditions:

1. $0 \leq WD^2{}_{P\boldsymbol{\mathcal{H}}F}(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{H}}') \leq 1$
2. $WD^2{}_{P\boldsymbol{\mathcal{H}}F}(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{H}}') = WD^2{}_{P\boldsymbol{\mathcal{H}}F}(\boldsymbol{\mathcal{H}}',\boldsymbol{\mathcal{H}})$
3. $WD^2{}_{P\boldsymbol{\mathcal{H}}F}(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{H}}') = 1 \Leftrightarrow \boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{H}}'$

*Remarks 2***:** If $w = (\frac{1}{M},\frac{1}{M},\dots,\frac{1}{M})^T$ then $WD^2{}_{P\boldsymbol{\mathcal{H}}F}(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{H}}') = D^2{}_{P\boldsymbol{\mathcal{H}}F}(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{H}}')$.

*Definition 10***:** A DSM for PHFSs $D^3{}_{P\boldsymbol{\mathcal{H}}F}(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{H}}')$ is of the form

$D^3{}_{P\boldsymbol{\mathcal{H}}F}(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{H}}')$

$$
= \frac{\sum_{i=1}^{M} 2\left(\frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l}\mathfrak{M}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\mathfrak{M}_{\boldsymbol{\mathcal{H}}'}^{j}(\mathfrak{x}_i)+\frac{1}{\mathfrak{A}}\sum_{j=1}^{l}\mathfrak{A}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\mathfrak{A}_{\boldsymbol{\mathcal{H}}'}^{j}(\mathfrak{x}_i)+\frac{1}{\mathfrak{N}}\sum_{j=1}^{l}N_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)N_{\boldsymbol{\mathcal{H}}'}^{j}(\mathfrak{x}_i)\right)}{\begin{array}{l}\sum_{i=1}^{M}\left(\frac{1}{L_{\mathfrak{M}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{N}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2\right)+\\ \sum_{i=1}^{M}\left(\frac{1}{L_{\mathfrak{M}_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\boldsymbol{\mathcal{H}}'}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\boldsymbol{\mathcal{H}}'}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{N}_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\boldsymbol{\mathcal{H}}'}^{j}(\mathfrak{x}_i)\right)^2\right)\end{array}} \quad (5)
$$

Which satisfy following conditions:

1. $0 \leq D^3{}_{P\boldsymbol{\mathcal{H}}F}(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{H}}') \leq 1$
2. $D^3{}_{P\boldsymbol{\mathcal{H}}F}(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{H}}') = D^3{}_{P\boldsymbol{\mathcal{H}}F}(\boldsymbol{\mathcal{H}}',\boldsymbol{\mathcal{H}})$
3. $D^3{}_{P\boldsymbol{\mathcal{H}}F}(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{H}}') = 1 \Leftrightarrow \boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{H}}'$.

Definition 11: A WDSM for PHFSs $WD^3{}_{P\boldsymbol{\mathcal{H}}F}(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{H}}')$ is of the form

$WD^3{}_{P\boldsymbol{\mathcal{H}}F}(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{H}}')$

$$
= \frac{\sum_{i=1}^{M} 2w_i^2\left(\frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l}\mathfrak{M}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\mathfrak{M}_{\boldsymbol{\mathcal{H}}'}^{j}(\mathfrak{x}_i)+\frac{1}{\dot{\mathfrak{A}}}\sum_{j=1}^{l}\mathfrak{A}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\mathfrak{A}_{\boldsymbol{\mathcal{H}}'}^{j}(\mathfrak{x}_i)+\frac{1}{\dot{\mathfrak{N}}}\sum_{j=1}^{l}N_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)N_{\boldsymbol{\mathcal{H}}'}^{j}(\mathfrak{x}_i)\right)}{\begin{array}{l}\sum_{i=1}^{M} w_i^2\left(\frac{1}{L_{\mathfrak{M}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{N}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2\right)+\\ \sum_{i=1}^{M} w_i^2\left(\frac{1}{L_{\mathfrak{M}_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\boldsymbol{\mathcal{H}}'}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\boldsymbol{\mathcal{H}}'}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{N}_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\boldsymbol{\mathcal{H}}'}^{j}(\mathfrak{x}_i)\right)^2\right)\end{array}} \quad (6)
$$

Which satisfy following conditions:

1. $0 \leq WD^3{}_{P\boldsymbol{\mathcal{H}}F}(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{H}}') \leq 1$
2. $WD^3{}_{P\boldsymbol{\mathcal{H}}F}(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{H}}') = WD^3{}_{P\boldsymbol{\mathcal{H}}F}(\boldsymbol{\mathcal{H}}',\boldsymbol{\mathcal{H}})$
3. $WD^3{}_{P\boldsymbol{\mathcal{H}}F}(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{H}}') = 1 \Leftrightarrow \boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{H}}'$

Remarks 3: If $w = (\frac{1}{M},\frac{1}{M},\dots,\frac{1}{M})^T$ then $WD^3{}_{P\boldsymbol{\mathcal{H}}F}(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{H}}') = D^3{}_{P\boldsymbol{\mathcal{H}}F}(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{H}}')$.

Definition 12: A DSM for PHFSs $D^4{}_{P\boldsymbol{\mathcal{H}}F}(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{H}}')$ is of the form

$D^4{}_{P\boldsymbol{\mathcal{H}}F}(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{H}}')$

$$
= \frac{2\sum_{i=1}^{M}\left(\begin{array}{c}\frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l}\mathfrak{M}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\mathfrak{M}_{\boldsymbol{\mathcal{H}}'}^{j}(\mathfrak{x}_i)+\frac{1}{\dot{\mathfrak{A}}}\sum_{j=1}^{l}\mathfrak{A}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\mathfrak{A}_{\boldsymbol{\mathcal{H}}'}^{j}(\mathfrak{x}_i)+\\ \frac{1}{\dot{\mathfrak{N}}}\sum_{j=1}^{l}N_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)N_{\boldsymbol{\mathcal{H}}'}^{j}(\mathfrak{x}_i)+\frac{1}{\dot{\pi}}\sum_{j=1}^{l}\pi_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\pi_{\boldsymbol{\mathcal{H}}'}^{j}(\mathfrak{x}_i)\end{array}\right)}{\begin{array}{l}\sum_{i=1}^{M}\left(\begin{array}{c}\frac{1}{L_{\mathfrak{M}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2+\\ \frac{1}{L_{\mathfrak{N}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\pi_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\pi_{\boldsymbol{\mathcal{H}}}^{j}(\mathfrak{x}_i)\right)^2\end{array}\right)+\\ \sum_{i=1}^{M}\left(\begin{array}{c}\frac{1}{L_{\mathfrak{M}_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\boldsymbol{\mathcal{H}}'}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\boldsymbol{\mathcal{H}}'}^{j}(\mathfrak{x}_i)\right)^2+\\ \frac{1}{L_{\mathfrak{N}_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\boldsymbol{\mathcal{H}}'}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\pi_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\pi_{\boldsymbol{\mathcal{H}}'}^{j}(\mathfrak{x}_i)\right)^2\end{array}\right)\end{array}} \quad (7)
$$

Which satisfy following conditions:

1. $0 \leq D^4{}_{P\boldsymbol{\mathcal{H}}F}(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{H}}') \leq 1$

2. $D^4{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') = D^4{}_{P\mathcal{H}F}(\mathcal{H}',\mathcal{H})$
3. $D^4{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') = 1 \Leftrightarrow \mathcal{H} = \mathcal{H}'$.

Definition 12: A WDSM for PHFSs $WD^4{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}')$ is of the form

$$WD^4{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}')$$

$$= \frac{2\sum_{i=1}^{M} w_i^2 \begin{pmatrix} \frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l} \mathfrak{M}_{\mathcal{H}}^j(\mathfrak{x}_i)\mathfrak{M}_{\mathcal{H}'}^j(\mathfrak{x}_i) + \frac{1}{\dot{\mathfrak{A}}}\sum_{j=1}^{l} \mathfrak{A}_{\mathcal{H}}^j(\mathfrak{x}_i)\mathfrak{A}_{\mathcal{H}'}^j(\mathfrak{x}_i) + \\ \frac{1}{\dot{\mathfrak{N}}}\sum_{j=1}^{l} N_{\mathcal{H}}^j(\mathfrak{x}_i)N_{\mathcal{H}'}^j(\mathfrak{x}_i) + \frac{1}{\dot{\pi}}\sum_{j=1}^{l} \pi_{\mathcal{H}}^j(\mathfrak{x}_i)\pi_{\mathcal{H}'}^j(\mathfrak{x}_i) \end{pmatrix}}{\begin{matrix}\sum_{i=1}^{M} w_i^2 \begin{pmatrix} \frac{1}{L_{\mathfrak{M}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^j(\mathfrak{x}_i)\right)^2 + \frac{1}{L_{\mathfrak{A}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}}^j(\mathfrak{x}_i)\right)^2 + \\ \frac{1}{L_{\mathfrak{N}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}}^j(\mathfrak{x}_i)\right)^2 + \frac{1}{L_{\pi_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\pi_{\mathcal{H}}^j(\mathfrak{x}_i)\right)^2 \end{pmatrix} + \\ \sum_{i=1}^{M} w_i^2 \begin{pmatrix} \frac{1}{L_{\mathfrak{M}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}'}^j(\mathfrak{x}_i)\right)^2 + \frac{1}{L_{\mathfrak{A}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}'}^j(\mathfrak{x}_i)\right)^2 + \\ \frac{1}{L_{\mathfrak{N}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}'}^j(\mathfrak{x}_i)\right)^2 + \frac{1}{L_{\pi_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\pi_{\mathcal{H}'}^j(\mathfrak{x}_i)\right)^2 \end{pmatrix}\end{matrix}} \quad (8)$$

Which satisfy the following conditions:

1. $0 \le WD^4{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') \le 1$
2. $WD^4{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') = WD^4{}_{P\mathcal{H}F}(\mathcal{H}',\mathcal{H})$
3. $WD^4{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') = 1 \Leftrightarrow \mathcal{H} = \mathcal{H}'$

Remarks 4: If we take $w = (\frac{1}{M}, \frac{1}{M}, \dots, \frac{1}{M})^T$ then $WD^4{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') = D^4{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}')$.

IV. THE GENERALIZED DICE SIMILARITY MEASURE FOR PICTURE HESITANT FUZZY SET

In this section, we generalized the DSM and WDSM for PHFS and also discussed its special cases is known as projection/asymmetric measures for PHFS. Throughout this manuscript $0 \le \rho \le 1$.

Definition 13: A GDSM for PHFSs $GD^1{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}')$ is of the form

$$GD^1{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}')$$

$$= \frac{1}{M}\sum_{i=1}^{M} \frac{\begin{pmatrix} \frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l} \mathfrak{M}_{\mathcal{H}}^j(\mathfrak{x}_i)\mathfrak{M}_{\mathcal{H}'}^j(\mathfrak{x}_i) + \frac{1}{\dot{\mathfrak{A}}}\sum_{j=1}^{l} \mathfrak{A}_{\mathcal{H}}^j(\mathfrak{x}_i)\mathfrak{A}_{\mathcal{H}'}^j(\mathfrak{x}_i) + \\ \frac{1}{\dot{\mathfrak{N}}}\sum_{j=1}^{l} N_{\mathcal{H}}^j(\mathfrak{x}_i)N_{\mathcal{H}'}^j(\mathfrak{x}_i) \end{pmatrix}}{\begin{pmatrix} \gamma \begin{pmatrix} \frac{1}{L_{\mathfrak{M}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^j(\mathfrak{x}_i)\right)^2 + \frac{1}{L_{\mathfrak{A}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}}^j(\mathfrak{x}_i)\right)^2 + \\ \frac{1}{L_{\mathfrak{N}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}}^j(\mathfrak{x}_i)\right)^2 \end{pmatrix} + \\ (1-\gamma) \begin{pmatrix} \frac{1}{L_{\mathfrak{M}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}'}^j(\mathfrak{x}_i)\right)^2 + \frac{1}{L_{\mathfrak{A}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}'}^j(\mathfrak{x}_i)\right)^2 + \\ \frac{1}{L_{\mathfrak{N}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}'}^j(\mathfrak{x}_i)\right)^2 \end{pmatrix} \end{pmatrix}} \quad (9)$$

Which satisfy following conditions:

1. $0 \leq GD^1{}_{P\mathcal{H}F}(\mathcal{H}, \mathcal{H}') \leq 1$
2. $GD^1{}_{P\mathcal{H}F}(\mathcal{H}, \mathcal{H}') = GD^1{}_{P\mathcal{H}F}(\mathcal{H}', \mathcal{H})$
3. $GD^1{}_{P\mathcal{H}F}(\mathcal{H}, \mathcal{H}') = 1 \Leftrightarrow \mathcal{H} = \mathcal{H}'$

Definition 14: A WGDSM for PHFSs $WGD^1{}_{P\mathcal{H}F}(\mathcal{H}, \mathcal{H}')$ is of the form

$WGD^1{}_{P\mathcal{H}F}(\mathcal{H}, \mathcal{H}')$

$$= \sum_{i=1}^{M} w_i \frac{\left(\begin{array}{c}\frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l} \mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i) + \frac{1}{\dot{\mathfrak{A}}}\sum_{j=1}^{l} \mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i) + \\ \frac{1}{\dot{\mathfrak{N}}}\sum_{j=1}^{l} N_{\mathcal{H}}^{j}(\mathfrak{x}_i)N_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\end{array}\right)}{\left(\begin{array}{c}\gamma\left(\begin{array}{c}\frac{1}{L_{\mathfrak{M}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2 + \frac{1}{L_{\mathfrak{A}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2 + \\ \frac{1}{L_{\mathfrak{N}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2\end{array}\right) + \\ (1-\gamma)\left(\begin{array}{c}\frac{1}{L_{\mathfrak{M}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2 + \frac{1}{L_{\mathfrak{A}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2 + \\ \frac{1}{L_{\mathfrak{N}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2\end{array}\right)\end{array}\right)} \tag{10}$$

Which satisfy following conditions:

1. $0 \leq WGD^1{}_{P\mathcal{H}F}(\mathcal{H}, \mathcal{H}') \leq 1$
2. $WGD^1{}_{P\mathcal{H}F}(\mathcal{H}, \mathcal{H}') = WGD^1{}_{P\mathcal{H}F}(\mathcal{H}', \mathcal{H})$
3. $WGD^1{}_{P\mathcal{H}F}(\mathcal{H}, \mathcal{H}') = 1 \Leftrightarrow \mathcal{H} = \mathcal{H}'$

Remarks 5: If $w = (\frac{1}{M}, \frac{1}{M}, \dots, \frac{1}{M})^T$ then $WGD^1{}_{P\mathcal{H}F}(\mathcal{H}, \mathcal{H}') = GD^1{}_{P\mathcal{H}F}(\mathcal{H}, \mathcal{H}')$.

Definition 15: A GDSM for PHFSs $GD^2{}_{P\mathcal{H}F}(\mathcal{H}, \mathcal{H}')$ is of the form

$GD^2{}_{P\mathcal{H}F}(\mathcal{H}, \mathcal{H}')$

$$= \frac{1}{M}\sum_{i=1}^{M} \frac{\left(\begin{array}{c}\frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l} \mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i) + \frac{1}{\dot{\mathfrak{A}}}\sum_{j=1}^{l} \mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i) + \\ \frac{1}{\dot{\mathfrak{N}}}\sum_{j=1}^{l} N_{\mathcal{H}}^{j}(\mathfrak{x}_i)N_{\mathcal{H}'}^{j}(\mathfrak{x}_i) + \frac{1}{\dot{\pi}}\sum_{j=1}^{l} \pi_{\mathcal{H}}^{j}(\mathfrak{x}_i)\pi_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\end{array}\right)}{\left(\begin{array}{c}\gamma\left(\begin{array}{c}\frac{1}{L_{\mathfrak{M}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2 + \frac{1}{L_{\mathfrak{A}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2 + \\ \frac{1}{L_{\mathfrak{N}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2 + \frac{1}{L_{\pi_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\pi_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2\end{array}\right) + \\ (1-\gamma)\left(\begin{array}{c}\frac{1}{L_{\mathfrak{M}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2 + \frac{1}{L_{\mathfrak{A}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2 + \\ \frac{1}{L_{\mathfrak{N}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2 + \frac{1}{L_{\pi_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\pi_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2\end{array}\right)\end{array}\right)} \tag{11}$$

Which satisfy following conditions:

1. $0 \leq GD^2{}_{P\mathcal{H}F}(\mathcal{H}, \mathcal{H}') \leq 1$
2. $GD^2{}_{P\mathcal{H}F}(\mathcal{H}, \mathcal{H}') = GD^2{}_{P\mathcal{H}F}(\mathcal{H}', \mathcal{H})$
3. $GD^2{}_{P\mathcal{H}F}(\mathcal{H}, \mathcal{H}') = 1 \Leftrightarrow \mathcal{H} = \mathcal{H}'$

Definition 16: A WGDSM for PHFSs $WGD^2{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}')$ is of the form

$WGD^2{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}')$

$$= \sum_{i=1}^{M} w_i \frac{2\left(\begin{array}{c}\frac{1}{\mathfrak{M}}\sum_{j=1}^{l}\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\frac{1}{\mathfrak{A}}\sum_{j=1}^{l}\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\\ \frac{1}{\mathfrak{N}}\sum_{j=1}^{ls}N_{\mathcal{H}}^{j}(\mathfrak{x}_i)N_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\frac{1}{\pi}\sum_{j=1}^{l}\pi_{\mathcal{H}}^{j}(\mathfrak{x}_i)\pi_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\end{array}\right)}{\left(\begin{array}{c}\gamma\left(\begin{array}{c}\frac{1}{L_{\mathfrak{M}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\\ \frac{1}{L_{\mathfrak{N}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\pi_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\pi_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2\end{array}\right)+\\ (1-\gamma)\left(\begin{array}{c}\frac{1}{L_{\mathfrak{M}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2+\\ \frac{1}{L_{\mathfrak{N}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\pi_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\pi_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2\end{array}\right)\end{array}\right)} \quad (12)$$

Which satisfy following conditions:

1. $0 \le WGD^2{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') \le 1$
2. $WGD^2{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') = WGD^2{}_{P\mathcal{H}F}(\mathcal{H}',\mathcal{H})$
3. $WGD^2{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') = 1 \Leftrightarrow \mathcal{H} = \mathcal{H}'$

Remarks 6: If $w = (\frac{1}{M},\frac{1}{M},\ldots,\frac{1}{M})^T$ then $WD^2{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') = D^2{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}')$.

Definition 17: A GDSM for PHFSs $GD^3{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}')$ is of the form

$GD^3{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}')$

$$= \frac{\sum_{i=1}^{M}\left(\frac{1}{\mathfrak{M}}\sum_{j=1}^{l}\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\frac{1}{\mathfrak{A}}\sum_{j=1}^{l}\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\frac{1}{\mathfrak{N}}\sum_{j=1}^{l}N_{\mathcal{H}}^{j}(\mathfrak{x}_i)N_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)}{\begin{array}{c}\gamma\sum_{i=1}^{M}\left(\frac{1}{L_{\mathfrak{M}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{N}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2\right)+\\ (1-\gamma)\sum_{i=1}^{M}\left(\begin{array}{c}\frac{1}{L_{\mathfrak{M}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2+\\ \frac{1}{L_{\mathfrak{N}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2\end{array}\right)\end{array}} \quad (13)$$

Which satisfy following conditions:

1. $0 \le GD^3{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') \le 1$
2. $GD^3{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') = GD^3{}_{P\mathcal{H}F}(\mathcal{H}',\mathcal{H})$
3. $GD^3{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') = 1 \Leftrightarrow \mathcal{H} = \mathcal{H}'$.

Definition 18: A WGDSM for PHFSs $WGD^3{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}')$ is of the form

$WGD^3{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}')$

$$=\frac{\sum_{i=1}^{M} w_i^2\left(\frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l}\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\frac{1}{\dot{\mathfrak{A}}}\sum_{j=1}^{l}\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\frac{1}{\dot{\mathfrak{N}}}\sum_{j=1}^{l}N_{\mathcal{H}}^{j}(\mathfrak{x}_i)N_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)}{\gamma\sum_{i=1}^{M} w_i^2\left(\frac{1}{L_{\mathfrak{M}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{N}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2\right)+(1-\gamma)\sum_{i=1}^{M} w_i^2\left(\frac{1}{L_{\mathfrak{M}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{N}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2\right)} \quad (14)$$

Which satisfy following conditions:

1. $0 \leq WGD^3{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') \leq 1$
2. $WGD^3{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') = WGD^3{}_{P\mathcal{H}F}(\mathcal{H}',\mathcal{H})$
3. $WGD^3{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') = 1 \Leftrightarrow \mathcal{H} = \mathcal{H}'$

*Remarks 7***:** If $w = (\frac{1}{M},\frac{1}{M},\dots,\frac{1}{M})^T$ then $WGD^3{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') = GD^3{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}')$.

*Definition 12***:** A GDSM for PHFSs $GD^4{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}')$ is of the form

$GD^4{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}')$

$$=\frac{\sum_{i=1}^{M}\left(\frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l}\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\frac{1}{\dot{\mathfrak{A}}}\sum_{j=1}^{l}\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\frac{1}{\dot{\mathfrak{N}}}\sum_{j=1}^{l}N_{\mathcal{H}}^{j}(\mathfrak{x}_i)N_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\frac{1}{\dot{\pi}}\sum_{j=1}^{l}\pi_{\mathcal{H}}^{j}(\mathfrak{x}_i)\pi_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)}{\gamma\sum_{i=1}^{M}\left(\frac{1}{L_{\mathfrak{M}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{N}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\pi_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\pi_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2\right)+(1-\gamma)\sum_{i=1}^{M}\left(\frac{1}{L_{\mathfrak{M}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{N}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\pi_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\pi_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2\right)} \quad (15)$$

Which satisfy following conditions:

1. $0 \leq GD^4{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') \leq 1$
2. $GD^4{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') = GD^4{}_{P\mathcal{H}F}(\mathcal{H}',\mathcal{H})$
3. $GD^4{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}') = 1 \Leftrightarrow \mathcal{H} = \mathcal{H}'$.

*Definition 12***:** A WGDSM for PHFSs $WGD^4{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}')$ is of the form

$WGD^4{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}')$

$$= \frac{\sum_{i=1}^{M} w_i^2 \begin{pmatrix} \frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l} \mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i) + \frac{1}{\dot{\mathfrak{A}}}\sum_{j=1}^{l} \mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i) + \\ \frac{1}{\dot{\mathfrak{N}}}\sum_{j=1}^{l} N_{\mathcal{H}}^{j}(\mathfrak{x}_i)N_{\mathcal{H}'}^{j}(\mathfrak{x}_i) + \frac{1}{\dot{\pi}}\sum_{j=1}^{l} \pi_{\mathcal{H}}^{j}(\mathfrak{x}_i)\pi_{\mathcal{H}'}^{j}(\mathfrak{x}_i) \end{pmatrix}}{\gamma\sum_{i=1}^{M} w_i^2 \begin{pmatrix} \frac{1}{L_{\mathfrak{M}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2 + \frac{1}{L_{\mathfrak{A}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2 + \\ \frac{1}{L_{\mathfrak{N}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2 + \frac{1}{L_{\pi_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\pi_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2 \end{pmatrix} + (1-\gamma)\sum_{i=1}^{M} w_i^2 \begin{pmatrix} \frac{1}{L_{\mathfrak{M}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2 + \frac{1}{L_{\mathfrak{A}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2 + \\ \frac{1}{L_{\mathfrak{N}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2 + \frac{1}{L_{\pi_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\pi_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2 \end{pmatrix}} \quad (16)$$

Which satisfy the following conditions:

1. $0 \leq WGD^4{}_{P\mathcal{H}F}(\mathcal{H}, \mathcal{H}') \leq 1$
2. $WGD^4{}_{P\mathcal{H}F}(\mathcal{H}, \mathcal{H}') = WGD^4{}_{P\mathcal{H}F}(\mathcal{H}', \mathcal{H})$
3. $WGD^4{}_{P\mathcal{H}F}(\mathcal{H}, \mathcal{H}') = 1 \Leftrightarrow \mathcal{H} = \mathcal{H}'$

Remarks 8: If we take $w = (\frac{1}{M}, \frac{1}{M}, \ldots, \frac{1}{M})^T$ then $WGD^4{}_{P\mathcal{H}F}(\mathcal{H}, \mathcal{H}') = GD^4{}_{P\mathcal{H}F}(\mathcal{H}, \mathcal{H}')$.

4.1. Special Cases

Here we are discussing some special cases of GDSM and WGDSM for PHFSs for different values of γ. For $\gamma = 0$ eq. (9) becomes

$GD^1{}_{P\mathcal{H}F}(\mathcal{H}, \mathcal{H}')$

$$= \frac{1}{M}\sum_{i=1}^{M} \frac{\begin{pmatrix} \frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l} \mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i) + \frac{1}{\dot{\mathfrak{A}}}\sum_{j=1}^{l} \mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i) + \\ \frac{1}{\dot{\mathfrak{N}}}\sum_{j=1}^{l} N_{\mathcal{H}}^{j}(\mathfrak{x}_i)N_{\mathcal{H}'}^{j}(\mathfrak{x}_i) \end{pmatrix}}{\begin{pmatrix} 0\begin{pmatrix} \frac{1}{L_{\mathfrak{M}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2 + \frac{1}{L_{\mathfrak{A}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2 + \\ \frac{1}{L_{\mathfrak{N}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2 \end{pmatrix} + \\ (1-0)\begin{pmatrix} \frac{1}{L_{\mathfrak{M}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2 + \frac{1}{L_{\mathfrak{A}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2 + \\ \frac{1}{L_{\mathfrak{N}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2 \end{pmatrix} \end{pmatrix}}$$

$$= \frac{1}{M}\sum_{i=1}^{M} \frac{\begin{pmatrix} \frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l} \mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i) + \frac{1}{\dot{\mathfrak{A}}}\sum_{j=1}^{l} \mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i) + \\ \frac{1}{\dot{\mathfrak{N}}}\sum_{j=1}^{l} N_{\mathcal{H}}^{j}(\mathfrak{x}_i)N_{\mathcal{H}'}^{j}(\mathfrak{x}_i) \end{pmatrix}}{\begin{pmatrix} \frac{1}{L_{\mathfrak{M}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2 + \frac{1}{L_{\mathfrak{A}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2 + \\ \frac{1}{L_{\mathfrak{N}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2 \end{pmatrix}} \quad (17)$$

Similarly for $\gamma = 0.5$,

$GD^1{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}')$

$$=\frac{1}{M}\sum_{i=1}^{M}\frac{\left(\begin{array}{c}\frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l}\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\frac{1}{\dot{\mathfrak{A}}}\sum_{j=1}^{l}\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\\ \frac{1}{\dot{\mathfrak{N}}}\sum_{j=1}^{l}N_{\mathcal{H}}^{j}(\mathfrak{x}_i)N_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\end{array}\right)}{\left(\begin{array}{c}0.5\left(\begin{array}{c}\frac{1}{L_{\mathfrak{M}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\\ \frac{1}{L_{\mathfrak{N}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2\end{array}\right)+\\ (1-0.5)\left(\begin{array}{c}\frac{1}{L_{\mathfrak{M}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2+\\ \frac{1}{L_{\mathfrak{N}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2\end{array}\right)\end{array}\right)}$$

$$=\frac{1}{M}\sum_{i=1}^{M}\frac{2\left(\begin{array}{c}\frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l}\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\frac{1}{\dot{\mathfrak{A}}}\sum_{j=1}^{l}\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\\ \frac{1}{\dot{\mathfrak{N}}}\sum_{j=1}^{l}N_{\mathcal{H}}^{j}(\mathfrak{x}_i)N_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\end{array}\right)}{\left(\begin{array}{c}\left(\begin{array}{c}\frac{1}{L_{\mathfrak{M}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\\ \frac{1}{L_{\mathfrak{N}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2\end{array}\right)+\\ \left(\begin{array}{c}\frac{1}{L_{\mathfrak{M}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2+\\ \frac{1}{L_{\mathfrak{N}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2\end{array}\right)\end{array}\right)}=D^1{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}')$$

Again we check for $\gamma = 0.5$,

$GD^1{}_{P\mathcal{H}F}(\mathcal{H},\mathcal{H}')$

$$=\frac{1}{M}\sum_{i=1}^{M}\frac{\left(\begin{array}{c}\frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l}\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\frac{1}{\dot{\mathfrak{A}}}\sum_{j=1}^{l}\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\\ \frac{1}{\dot{\mathfrak{N}}}\sum_{j=1}^{l}N_{\mathcal{H}}^{j}(\mathfrak{x}_i)N_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\end{array}\right)}{\left(\begin{array}{c}1\left(\begin{array}{c}\frac{1}{L_{\mathfrak{M}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\\ \frac{1}{L_{\mathfrak{N}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2\end{array}\right)+\\ (1-1)\left(\begin{array}{c}\frac{1}{L_{\mathfrak{M}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2+\\ \frac{1}{L_{\mathfrak{N}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)^2\end{array}\right)\end{array}\right)}$$

$$= \frac{1}{M}\sum_{i=1}^{M}\frac{\left(\begin{array}{c}\frac{1}{\mathfrak{M}}\sum_{j=1}^{l}\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\frac{1}{\mathfrak{A}}\sum_{j=1}^{l}\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\\ \frac{1}{\mathfrak{N}}\sum_{j=1}^{l}N_{\mathcal{H}}^{j}(\mathfrak{x}_i)N_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\end{array}\right)}{\left(\begin{array}{c}\frac{1}{L_{\mathfrak{M}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{A}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\\ \frac{1}{L_{\mathfrak{N}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2\end{array}\right)} \quad (18)$$

In the above discussion we concluded that if we take the value of $\gamma = 0$ or 1, then the Eq. (9) reduce into Eq. (17) and Eq. (18), which is called asymmetric/projection measures for PHFS. Similarly, we investigated the Eq. (10-16) for $\gamma = 0,0.5,1$.

V. MULTI-ATTRIBUTE DECISION MAKING

In this section, we will proposed the application of building material recognition for WGDSM in the environment of PHFS. The application is taken from ref. [28].

5.2. Numerical Example

In this section, we developed the example of building material recognition using the WGDSM for PHFS. The example is solve by the four different types of notions like Eq. (10), Eq. (12), Eq. (14) and Eq. (16), and the building material is of the form: sealant, floor varnish, wall paint and polyvinyl chloride flooring, which is denoted the PHFSs $\mathcal{H}_i$ in space $X = \{\mathfrak{x}_1, \mathfrak{x}_2, \dots, \mathfrak{x}_6\}$, with weight vector $w = (0.14,0.18,0.16,0.14,0.18,0.2)^T$ so we are construct the matrix of PHFSs for comparing the values of WGDSM for PHFSs. Now, we consider

	$\mathcal{H}_1$	$\mathcal{H}_2$	$\mathcal{H}_3$	$\mathcal{H}$
$\mathfrak{x}_1$	$\left(\begin{array}{c}\{0.3,0.4\},\\ \{0.2,0.1\},\{0.4,0.3\}\end{array}\right)$	$\left(\begin{array}{c}\{0.7,0.4\},\\ \{0.2,0.1\},\{0.1,0.1\}\end{array}\right)$	$\left(\begin{array}{c}\{0.2,0.3\},\\ \{0.3,0.3\},\{0.4,0.3\}\end{array}\right)$	$\left(\begin{array}{c}\{0.1,0.2\},\\ \{0.2,0.3\},\{0.3,0.4\}\end{array}\right)$
$\mathfrak{x}_2$	$\left(\begin{array}{c}\{0.1,0.3\},\\ \{0.3,0.1\},\{0.3,0.3\}\end{array}\right)$	$\left(\begin{array}{c}\{0.6,0.3\},\\ \{0.2,0.1\},\{0.2,0.2\}\end{array}\right)$	$\left(\begin{array}{c}\{0.8,0.8\},\\ \{01,0.1\},\{0.1,0.1\}\end{array}\right)$	$\left(\begin{array}{c}\{0.2,0.2\},\\ \{0.4,0.1\},\{0.4,0.3\}\end{array}\right)$
$\mathfrak{x}_3$	$\left(\begin{array}{c}\{0.3,0.2\},\\ \{0.2,0.2\},\{0.3,0.1\}\end{array}\right)$	$\left(\begin{array}{c}\{0.1,0.2\},\\ \{0.3,0.3\},\{0.3,0.5\}\end{array}\right)$	$\left(\begin{array}{c}\{0.1,0.1\},\\ \{0.4,0.4\},\{0.5,0.5\}\end{array}\right)$	$\left(\begin{array}{c}\{0.2,0.2\},\\ \{0.3,0.3\},\{0.5,0.2\}\end{array}\right)$
$\mathfrak{x}_4$	$\left(\begin{array}{c}\{0.1,0.1\},\\ \{0.2,0.2\},\{0.4,0.4\}\end{array}\right)$	$\left(\begin{array}{c}\{0.2,0.2\},\\ \{0.3,0.3\},\{0.5,0.5\}\end{array}\right)$	$\left(\begin{array}{c}\{0.2,0.2\},\\ \{0.3,0.3\},\{0.4,0.4\}\end{array}\right)$	$\left(\begin{array}{c}\{0.6,0.6\},\\ \{0.3,0.2\},\{0.1,0.1\}\end{array}\right)$
$\mathfrak{x}_5$	$\left(\begin{array}{c}\{0.2,0.2\},\\ \{0.3,0.3\},\{0.5,0.5\}\end{array}\right)$	$\left(\begin{array}{c}\{0.4,0.3\},\\ \{0.2,0.1\},\{0.1,0.3\}\end{array}\right)$	$\left(\begin{array}{c}\{0.3,0.3\},\\ \{0.3,0.4\},\{0.2,0.2\}\end{array}\right)$	$\left(\begin{array}{c}\{0.5,0.5\},\\ \{0.4,0.4\},\{0.1,0.1\}\end{array}\right)$
$\mathfrak{x}_6$	$\left(\begin{array}{c}\{0.4,0.4\},\\ \{0.5,0.5\},\{0.1,0.1\}\end{array}\right)$	$\left(\begin{array}{c}\{0.2,0.3\},\\ \{0.3,0.1\},\{0.4,0.3\}\end{array}\right)$	$\left(\begin{array}{c}\{0.1,0.3\},\\ \{0.3,0.1\},\{0.3,0.3\}\end{array}\right)$	$\left(\begin{array}{c}\{0.8,0.6\},\\ \{0.1,0.1\},\{0.1,0.1\}\end{array}\right)$

Table 1(Numerical data of building material)

We consider the Eq. (10) to verify the above matrix.

γ	$D^1{}_{PHF}(\mathcal{H}_1, \mathcal{H})$	$D^1{}_{PHF}(\mathcal{H}_2, \mathcal{H})$	$D^1{}_{PHF}(\mathcal{H}_3, \mathcal{H})$	**Ranking**
0	0.696	0.659	0.772	$\mathcal{H}_3 > \mathcal{H}_1 > \mathcal{H}_2$

0.2	0.720	0.668	0.734	$\mathcal{H}_3 > \mathcal{H}_1 > \mathcal{H}_2$
0.5	0.768	0.698	0.722	$\mathcal{H}_1 > \mathcal{H}_3 > \mathcal{H}_2$
0.7	0.810	0.734	0.740	$\mathcal{H}_1 > \mathcal{H}_3 > \mathcal{H}_2$
1.0	0.897	0.828	0.819	$\mathcal{H}_1 > \mathcal{H}_2 > \mathcal{H}_3$

Table 2(Numerical data of building material)

We consider the Eq. (12) to verify the above matrix.

γ	$D^2{}_{P\mathcal{H}F}(\mathcal{H}_1, \mathcal{H})$	$D^2{}_{P\mathcal{H}F}(\mathcal{H}_2, \mathcal{H})$	$D^2{}_{P\mathcal{H}F}(\mathcal{H}_3, \mathcal{H})$	**Ranking**
0	0.597	0.548	0.686	$\mathcal{H}_3 > \mathcal{H}_1 > \mathcal{H}_2$
0.2	0.626	0.571	0.680	$\mathcal{H}_3 > \mathcal{H}_2 > \mathcal{H}_1$
0.5	0.678	0.615	0.692	$\mathcal{H}_3 > \mathcal{H}_1 > \mathcal{H}_2$
0.7	0.719	0.653	0.713	$\mathcal{H}_1 > \mathcal{H}_3 > \mathcal{H}_2$
1.0	0.796	0.734	0.767	$\mathcal{H}_1 > \mathcal{H}_3 > \mathcal{H}_2$

Table 3(Numerical data of building material)

We consider the Eq. (14) to verify the above matrix.

γ	$D^3{}_{P\mathcal{H}F}(\mathcal{H}_1, \mathcal{H})$	$D^3{}_{P\mathcal{H}F}(\mathcal{H}_2, \mathcal{H})$	$D^3{}_{P\mathcal{H}F}(\mathcal{H}_3, \mathcal{H})$	**Ranking**
0	0.661	0.607	0.669	$\mathcal{H}_3 > \mathcal{H}_1 > \mathcal{H}_2$
0.2	0.691	0.638	0.675	$\mathcal{H}_1 > \mathcal{H}_3 > \mathcal{H}_2$
0.5	0.742	0.690	0.686	$\mathcal{H}_1 > \mathcal{H}_2 > \mathcal{H}_3$
0.7	0.780	0.730	0.693	$\mathcal{H}_1 > \mathcal{H}_2 > \mathcal{H}_3$
1.0	0.845	0.799	0.704	$\mathcal{H}_1 > \mathcal{H}_2 > \mathcal{H}_3$

Table 4(Numerical data of building material)

We consider the Eq. (16) to verify the above matrix.

γ	$D^4{}_{P\mathcal{H}F}(\mathcal{H}_1, \mathcal{H})$	$D^4{}_{P\mathcal{H}F}(\mathcal{H}_2, \mathcal{H})$	$D^4{}_{P\mathcal{H}F}(\mathcal{H}_3, \mathcal{H})$	**Ranking**
0	0.669	0.595	0.633	$\mathcal{H}_1 > \mathcal{H}_3 > \mathcal{H}_2$
0.2	0.692	0.615	0.642	$\mathcal{H}_1 > \mathcal{H}_3 > \mathcal{H}_2$
0.5	0.728	0.648	0.655	$\mathcal{H}_1 > \mathcal{H}_2 > \mathcal{H}_3$
0.7	0.754	0.671	0.665	$\mathcal{H}_1 > \mathcal{H}_2 > \mathcal{H}_3$

1.0	0.797	0.710	0.679	$\boldsymbol{\mathcal{H}}_1 > \boldsymbol{\mathcal{H}}_2 > \boldsymbol{\mathcal{H}}_3$

Table 5(Numerical data of building material)

VI. Comparative Study

In this section by using different conditions we reduce the proposed work into existing work. Basically, we show that the GDSM and WGDSM for PHFS is effective and more flexible to other concepts, which discussed in below.

Case 1: If we consider $\mathfrak{M}^j_{\boldsymbol{\mathcal{H}}}(\mathfrak{x}_i)$, $\mathfrak{A}^j_{\boldsymbol{\mathcal{H}}}(\mathfrak{x}_i)$, $\mathfrak{N}^j_{\boldsymbol{\mathcal{H}}}(\mathfrak{x}_i)$ and $\mathfrak{M}^j_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x}_i)$, $\mathfrak{A}^j_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x}_i)$, $\mathfrak{N}^j_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x}_i)$ is the form of singleton set, then Eq. (9) and Eq. (10) will be converted for PFS, which is proposed by Coung [12].

$GD^1{}_{PFS}(\boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{H}}')$

$$= \frac{1}{M}\sum_{i=1}^{M} \frac{\begin{pmatrix} \frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l} \mathfrak{M}^j_{\boldsymbol{\mathcal{H}}}(\mathfrak{x}_i)\mathfrak{M}^j_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x}_i) + \frac{1}{\dot{\mathfrak{A}}}\sum_{j=1}^{l} \mathfrak{A}^j_{\boldsymbol{\mathcal{H}}}(\mathfrak{x}_i)\mathfrak{A}^j_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x}_i) + \\ \frac{1}{\dot{\mathfrak{N}}}\sum_{j=1}^{l} N^j_{\boldsymbol{\mathcal{H}}}(\mathfrak{x}_i)N^j_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x}_i) \end{pmatrix}}{\begin{pmatrix} \gamma\begin{pmatrix} \frac{1}{L_{\mathfrak{M}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}^j_{\boldsymbol{\mathcal{H}}}(\mathfrak{x}_i)\right)^2 + \frac{1}{L_{\mathfrak{A}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}^j_{\boldsymbol{\mathcal{H}}}(\mathfrak{x}_i)\right)^2 + \\ \frac{1}{L_{\mathfrak{N}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}^j_{\boldsymbol{\mathcal{H}}}(\mathfrak{x}_i)\right)^2 \end{pmatrix} + \\ (1-\gamma)\begin{pmatrix} \frac{1}{L_{\mathfrak{M}_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}^j_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x}_i)\right)^2 + \frac{1}{L_{\mathfrak{A}_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}^j_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x}_i)\right)^2 + \\ \frac{1}{L_{\mathfrak{N}_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}^j_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x}_i)\right)^2 \end{pmatrix} \end{pmatrix}} \quad (19)$$

$WGD^1{}_{PFS}(\boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{H}}')$

$$= \sum_{i=1}^{M} w_i \frac{\begin{pmatrix} \frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l} \mathfrak{M}^j_{\boldsymbol{\mathcal{H}}}(\mathfrak{x}_i)\mathfrak{M}^j_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x}_i) + \frac{1}{\dot{\mathfrak{A}}}\sum_{j=1}^{l} \mathfrak{A}^j_{\boldsymbol{\mathcal{H}}}(\mathfrak{x}_i)\mathfrak{A}^j_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x}_i) + \\ \frac{1}{\dot{\mathfrak{N}}}\sum_{j=1}^{l} N^j_{\boldsymbol{\mathcal{H}}}(\mathfrak{x}_i)N^j_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x}_i) \end{pmatrix}}{\begin{pmatrix} \gamma\begin{pmatrix} \frac{1}{L_{\mathfrak{M}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}^j_{\boldsymbol{\mathcal{H}}}(\mathfrak{x}_i)\right)^2 + \frac{1}{L_{\mathfrak{A}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}^j_{\boldsymbol{\mathcal{H}}}(\mathfrak{x}_i)\right)^2 + \\ \frac{1}{L_{\mathfrak{N}_{\boldsymbol{\mathcal{H}}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}^j_{\boldsymbol{\mathcal{H}}}(\mathfrak{x}_i)\right)^2 \end{pmatrix} + \\ (1-\gamma)\begin{pmatrix} \frac{1}{L_{\mathfrak{M}_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}^j_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x}_i)\right)^2 + \frac{1}{L_{\mathfrak{A}_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{A}^j_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x}_i)\right)^2 + \\ \frac{1}{L_{\mathfrak{N}_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}^j_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x}_i)\right)^2 \end{pmatrix} \end{pmatrix}} \quad (20)$$

Case 2: If we consider $\mathfrak{M}^j_{\boldsymbol{\mathcal{H}}}(\mathfrak{x}_i)$, $\mathfrak{N}^j_{\boldsymbol{\mathcal{H}}}(\mathfrak{x}_i)$ and $\mathfrak{M}^j_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x}_i)$, $\mathfrak{N}^j_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x}_i)$ is the form of HFS and $\mathfrak{A}^j_{\boldsymbol{\mathcal{H}}}(\mathfrak{x}_i) = \mathfrak{A}^j_{\boldsymbol{\mathcal{H}}'}(\mathfrak{x}_i) = 0$, then Eq. (9) and Eq. (10) will be converted for IHFS, which is proposed by Beg and Rashid [41].

$GD^1{}_{I\boldsymbol{\mathcal{H}}FS}(\boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{H}}')$

$$=\frac{1}{M}\sum_{i=1}^{M}\frac{\left(\frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l}\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\frac{1}{\dot{N}}\sum_{j=1}^{l}N_{\mathcal{H}}^{j}(\mathfrak{x}_i)N_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)}{\begin{pmatrix}\gamma\left(\frac{1}{L_{\mathfrak{M}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{N}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2\right)+\\(1-\gamma)\left(\frac{1}{L_{\mathfrak{M}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{N}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2\right)\end{pmatrix}} \quad (21)$$

$WGD^1{}_{PFS}(\mathcal{H},\mathcal{H}')$

$$=\sum_{i=1}^{M}w_i\frac{\left(\frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l}\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\frac{1}{\dot{N}}\sum_{j=1}^{l}N_{\mathcal{H}}^{j}(\mathfrak{x}_i)N_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)}{\begin{pmatrix}\gamma\left(\frac{1}{L_{\mathfrak{M}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{N}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2\right)+\\(1-\gamma)\left(\frac{1}{L_{\mathfrak{M}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{N}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2\right)\end{pmatrix}} \quad (22)$$

Case 3: If we consider $\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)$, $\mathfrak{N}_{\mathcal{H}}^{j}(\mathfrak{x}_i)$ and $\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)$, $\mathfrak{N}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)$ is the form of singleton set and $\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)=\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)=0$, then Eq. (9) and Eq. (10) will be converted for IFS, which is proposed by Atanassove [2].

$GD^1{}_{IFS}(\mathcal{H},\mathcal{H}')$

$$=\frac{1}{M}\sum_{i=1}^{M}\frac{\left(\frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l}\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\frac{1}{\dot{N}}\sum_{j=1}^{l}N_{\mathcal{H}}^{j}(\mathfrak{x}_i)N_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)}{\begin{pmatrix}\gamma\left(\frac{1}{L_{\mathfrak{M}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{N}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2\right)+\\(1-\gamma)\left(\frac{1}{L_{\mathfrak{M}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{N}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2\right)\end{pmatrix}} \quad (23)$$

$WGD^1{}_{IFS}(\mathcal{H},\mathcal{H}')$

$$=\sum_{i=1}^{M}w_i\frac{\left(\frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l}\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)+\frac{1}{\dot{N}}\sum_{j=1}^{l}N_{\mathcal{H}}^{j}(\mathfrak{x}_i)N_{\mathcal{H}'}^{j}(\mathfrak{x}_i)\right)}{\begin{pmatrix}\gamma\left(\frac{1}{L_{\mathfrak{M}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{N}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2\right)+\\(1-\gamma)\left(\frac{1}{L_{\mathfrak{M}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2+\frac{1}{L_{\mathfrak{N}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{N}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2\right)\end{pmatrix}} \quad (24)$$

Case 4: If we consider $\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)$ and $\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)$, is the form of HFS and $\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{x}_i)=\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)=0$, $\mathfrak{N}_{\mathcal{H}}^{j}(\mathfrak{x}_i)=\mathfrak{N}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)=0$ then Eq. (9) and Eq. (10) will be converted for HFS, which is proposed by Torra [21].

$GD^1{}_{\mathcal{H}FS}(\mathcal{H},\mathcal{H}')$

$$=\frac{1}{M}\sum_{i=1}^{M}\frac{\frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l}\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{x}_i)}{\begin{pmatrix}\gamma\left(\frac{1}{L_{\mathfrak{M}_{\mathcal{H}}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2\right)+\\(1-\gamma)\left(\frac{1}{L_{\mathfrak{M}_{\mathcal{H}'}(\mathfrak{x})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{x}_i)\right)^2\right)\end{pmatrix}} \quad (25)$$

$WGD^1{}_{\mathcal{H}FS}(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{H}'})$

$$=\sum_{i=1}^{M} w_i \frac{\frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l}\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{X}_i)\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{X}_i)}{\left(\begin{array}{c}\gamma\left(\frac{1}{L_{\mathfrak{M}_{\mathcal{H}}(\mathfrak{X})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{X}_i)\right)^2\right)+\\(1-\gamma)\left(\frac{1}{L_{\mathfrak{M}_{\mathcal{H}'}(\mathfrak{X})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{X}_i)\right)^2\right)\end{array}\right)} \quad (27)$$

Case 5: If we consider $\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{X}_i)$ and $\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{X}_i)$, is the form of singleton set and $\mathfrak{A}_{\mathcal{H}}^{j}(\mathfrak{X}_i)=\mathfrak{A}_{\mathcal{H}'}^{j}(\mathfrak{X}_i)=0$, $\mathfrak{N}_{\mathcal{H}}^{j}(\mathfrak{X}_i)=\mathfrak{N}_{\mathcal{H}'}^{j}(\mathfrak{X}_i)=0$ then Eq. (9) and Eq. (10) will be converted for FS, which is proposed by [1].

$GD^1{}_{\mathcal{H}FS}(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{H}'})$

$$=\frac{1}{M}\sum_{i=1}^{M} \frac{\frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l}\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{X}_i)\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{X}_i)}{\left(\begin{array}{c}\gamma\left(\frac{1}{L_{\mathfrak{M}_{\mathcal{H}}(\mathfrak{X})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{X}_i)\right)^2\right)+\\(1-\gamma)\left(\frac{1}{L_{\mathfrak{M}_{\mathcal{H}'}(\mathfrak{X})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{X}_i)\right)^2\right)\end{array}\right)} \quad (28)$$

$WGD^1{}_{\mathcal{H}FS}(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{H}'})$

$$=\sum_{i=1}^{M} w_i \frac{\frac{1}{\dot{\mathfrak{M}}}\sum_{j=1}^{l}\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{X}_i)\mathfrak{M}_{\mathcal{H}'}^{j}(\mathfrak{X}_i)}{\left(\begin{array}{c}\gamma\left(\frac{1}{L_{\mathfrak{M}_{\mathcal{H}}(\mathfrak{X})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{X}_i)\right)^2\right)+\\(1-\gamma)\left(\frac{1}{L_{\mathfrak{M}_{\mathcal{H}'}(\mathfrak{X})}}\sum_{j=1}^{l}\left(\mathfrak{M}_{\mathcal{H}}^{j}(\mathfrak{X}_i)\right)^2\right)\end{array}\right)} \quad (29)$$

In above four cases it is proved that our proposed work is more generalized than that of existing work. Our proposed work is able to solve all data given in environments of IFSs, PFSs , HFS, IHFSs but conversely the existing work is not able to solve the data given in PHFS.

6.2. Advantages

In this section, we shall discuss the advantages of proposed work over a existing work. Our proposed work can handle the data given in PFS, IHFS, IFS, HFS and FS but the existing work cannot handle the data given in PHFSs. Here we reduced PHFS into PFS, IHFS, IFS, HFS and FS.

The numerical data table for **Case 1**, which is PFS such that

	$\boldsymbol{\mathcal{H}_1}$	$\boldsymbol{\mathcal{H}_2}$	$\boldsymbol{\mathcal{H}_3}$	$\boldsymbol{\mathcal{H}}$
$\boldsymbol{\mathfrak{X}_1}$	(0.3,0.2,0.4)	(0.7,0.2,0.1)	(0.2,0.3,0.4)	(0.1,0.2,0.3)
$\boldsymbol{\mathfrak{X}_2}$	(0.1,0.3,0.3)	(0.6,0.2,0.2)	(0.8,0.1,0.1)	(0.2,0.4,0.4)
$\boldsymbol{\mathfrak{X}_3}$	(0.3,0.2,0.3)	(0.1,0.3,0.3)	(0.1,0.4,0.5)	(0.2,0.3,0.5)

$\mathfrak{X}_4$	(0.1,0.2,0.4)	(0.2,0.3,0.5)	(0.2,0.3,0.4)	(0.6,0.3,0.1)
$\mathfrak{X}_5$	(0.2,0.3,0.5)	(0.4,0.2,0.1)	(0.3,0.3,0.2)	(0.5,0.4,0.1)
$\mathfrak{X}_6$	(0.4,0.5,0.1)	(0.2,0.3,0.4)	(0.1,0.3,0.3)	(0.8,0.1,0.1)

Table 6. (Numerical data of building material)

The numerical data table for **Case 2**, which is IHFS such that

	$\mathcal{H}_1$	$\mathcal{H}_2$	$\mathcal{H}_3$	$\mathcal{H}$
$\mathfrak{X}_1$	({0.3,0.4},{0.4,0.3})	({0.7,0.4},{0.1,0.1})	({0.2,0.3},{0.4,0.3})	({0.1,0.2},{0.3,0.4})
$\mathfrak{X}_2$	({0.1,0.3},{0.3,0.3})	({0.6,0.3},{0.2,0.2})	({0.8,0.8},{0.1,0.1})	({0.2,0.2},{0.4,0.3})
$\mathfrak{X}_3$	({0.3,0.2},{0.3,0.1})	({0.1,0.2},{0.3,0.5})	({0.1,0.1},{0.5,0.5})	({0.2,0.2},{0.5,0.2})
$\mathfrak{X}_4$	({0.1,0.1},{0.4,0.4})	({0.2,0.2},{0.5,0.5})	({0.2,0.2},{0.4,0.4})	({0.6,0.6},{0.1,0.1})
$\mathfrak{X}_5$	({0.2,0.2},{0.5,0.5})	({0.4,0.3},{0.1,0.3})	({0.3,0.3},{0.2,0.2})	({0.5,0.5},{0.1,0.1})
$\mathfrak{X}_6$	({0.4,0.4},{0.1,0.1})	({0.2,0.3},{0.4,0.3})	({0.1,0.3},{0.3,0.3})	({0.8,0.6},{0.1,0.1})

Table 7. (Numerical data of building material)

The numerical data table for **Case 3**, which is IFS such that

	$\mathcal{H}_1$	$\mathcal{H}_2$	$\mathcal{H}_3$	$\mathcal{H}$
$\mathfrak{X}_1$	(0.3,0.4)	(0.7,0.1)	(0.2,0.4)	(0.1,0.3)
$\mathfrak{X}_2$	(0.1,0.3)	(0.6,0.2)	(0.8,0.1)	(0.2,0.4)
$\mathfrak{X}_3$	(0.3,0.3)	(0.1,0.3)	(0.1,0.5)	(0.2,0.5)
$\mathfrak{X}_4$	(0.1,0.4)	(0.2,0.5)	(0.2,0.4)	(0.6,0.1)
$\mathfrak{X}_5$	(0.2,0.5)	(0.4,0.1)	(0.3,0.2)	(0.5,0.1)
$\mathfrak{X}_6$	(0.4,0.1)	(0.2,0.4)	(0.1,0.3)	(0.8,0.1)

Table 8. (Numerical data of building material)

The numerical data table for **Case 4**, which is HFS such that

	$\mathcal{H}_1$	$\mathcal{H}_2$	$\mathcal{H}_3$	$\mathcal{H}$
$\mathfrak{X}_1$	{0.3,0.4}	{0.7,0.4}	{0.2,0.3}	{0.1,0.2}
$\mathfrak{X}_2$	{0.1,0.3}	{0.6,0.3}	{0.8,0.8}	{0.2,0.2}
$\mathfrak{X}_3$	{0.3,0.2}	{0.1,0.2}	{0.1,0.1}	{0.2,0.2}

$\mathfrak{X}_4$	{0.1,0.1}	{0.2,0.2}	{0.2,0.2}	{0.6,0.6}
$\mathfrak{X}_5$	{0.2,0.2}	{0.4,0.3}	{0.3,0.3}	{0.5,0.5}
$\mathfrak{X}_6$	{0.4,0.4}	{0.2,0.3}	{0.1,0.3}	{0.8,0.6}

Table 9. (Numerical data of building material)

The above all examples are proved that our proposed methods is more reliable and more effective than existing methods, which is discussed in section 6. It is clear that PFS, IHFS, IFS, HFS, FS all are the special cases of our proposed methods. Therefore, the proposed method is superior than existing methods.

VII. Conclusion

The PHFS is described by four functions representing the membership, abstinence, non-membership and refusal grade. The pre-existing work is not able to handle picture fuzzy hesitant fuzzy types of information in the environment of fuzzy set theory. So in this paper we proposed the other form of DSMs and WDSMs for PHFS, which taken from [26]. After that we proposed GDSMs and GWDSMs for PHFSs. we also discussed the some special cases of GDSMs and WGDSMs for PHFS is known as projection/asymmetric measures for PHFSs. Then, we construct an application of building material recognition and find the best alternative with the base of criteria. We also derived a comparison between proposed and pre-existing work and advantages of proposed work are also discussed.

In future, we will extend the notion of Complex q-rung orthopair fuzzy sets [29-35], T-spherical fuzzy sets [35-40], and etc. [41-46], to improve the quality of the research works.

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