Technical Journal, University of Engineering and Technology (UET) Taxila, Pakistan Vol. 25 No. 4-2020
ISSN:1813-1786 (Print) 2313-7770 (Online)

Parameterization of Triangle Groups Isomorphic to Least Simple Group A5 of Projective Symplectic Groups

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Abstract- In this paper, we investigate the influence of A5on PSp(4,2) and PSp(4,3) and find the triangle group isomorphic to smallest simple group A5 using the Character table of Projective Symplectic groups. Also discuss some interesting results by use of character tables of the given groups.

Keywords- Projective Symplectic groups, Simple groups, Character Table, Triangular Group.

AMS Mathematics Subject Classification (2010)- 20D06, 20D20, 20D40, 20D60 .

I. INTRODUCTION

In this paper we introduced the existence of smallest simple alternating group A_5 in Projective Symplectic group. All the notations and terminologies are standard and discussed in [1,2,3], [4,5] while [6] discussed the permutation representations of triangular groups. In [7] gave a technique how to construct representations of Symplectic groups of different degrees and then many other authors introduced and present extensions of smallest simple group in different finite groups. In [8], proved A_5 as a subgroup of ${}^2F_4(2)$. In [9], the two conjugate classes of A_5 within S_6. Recently, in [10] proved A_5 as a subgroup of S_7. [11] introduced the same result in his excellent work in the groups other than symmetric groups. By [12], A_5 is the smallest non-abelian simple group of order $2^2.3.5$ and the smallest non-solvable group containing four conjugacy classes. Now we find A_5 within $PSp\ (4,\ 2)$ and $PSp\ (4,\ 3)$ by using character table of these groups. In [13] the projective symplectic group simply denoted by PSp is the group obtained from the Symplectic group $Sp_n\ (q)$ on factoring by the scalar matrices PSp is simple except for

$$PS_{p2}(2) = S_3$$

$$PS_{p2}(3) = A_4$$

$$PSp_4(2) = S_6$$

In complex field, [14] studies local system of generators and associates finite unitary groups with Symplectic groups. By [15,16], A_5 is the smallest non-abelian and non-solvable group of order 60 containing 4 conjugacy classes. A group $\Delta\,(2,\,3,\,k)$ of the form

$$\Delta\,(2,\,3,\,k) = <a,\,b\,;\,a^2 = b^3 = (ab)^k = 1>$$

where k is any positive integer, is known As triangular group discussed in (see [17].[18]). For k = 5, it is isomorphic to A_5.

In [19], the parametrization of extended triangle groups for various values of k has been described it as subgroups of PSL(2, q). While, in [20] gives characterization of symplectic groups of degree 4 over locally finite fields of degree 2 which can be described in class of periodic groups.

Moreover, the role of duality triangular groups is vital in characterization of symplectic groups with homomorphic images of Alternating group A_5. In representational point of view of these groups can be studied in [21] s-duality of triangular groups and its modular animalises has been calculated in finite fields.

II. PRELIMINARIES

In this section we present some basic definitions and examples and then prove some theorems. In this paper our main focus is to find A_5 in Projective Symplectic group and verify some basic results.

The $PSp_{2n}(q)$ groups are simple except for $PSp_2(2)$, $PSp_2(3)$ and $PSp_4(2)$.

2.1. Definition: Let $\Delta(2,3,5)$ *be a triangular group if* $|a| = 2$, $|b| = 3$, *then* $|ab| = 5$, *i.e.,* $G = \{<a, b>: a^2 = 1 = b^3 = (ab)^5\}$.

2.2. Definition: The Projective Symplectic group $PSp_n(q)$ is the group obtained from the symplectic

group $Sp_n(q)$ on factoring by the scalar matrices contained in that group.
In [22], Sympletic group and its subgroups having commutators with influence can be studied.

2.3. Definition: A triangle group is a group that can be realized geometrically by sequences of reflections across the sides of a triangle. Each triangle group is the symmetry group of a tilling of the Euclidean plane, the sphere or the hyperbolic plane by congruent triangles, each one a fundamental domain for the action.

$\Delta(2, 3, k) = <a, b; a^2 = b^3 = (ab)^k = 1>$

Example: There exist two conjugate classes with in S_6 using character table of S_6 [6].
In order to prove the main results, we use some basic theorems.

2.4. Theorem: Let G be a Projective Symplectic group PSp (4, 2). Then there exist two non-conjugate classes of simple group isomorphic to A_5 involved by classes C_β, C_γ and C_α such that $\beta^2 = \gamma^3 = 1 = (\beta.\gamma) = \alpha^5$ in Psp (4, 2).

Proof: Let $G_2 \leq$ PSp (4, 2) be a subgroup isomorphic to A_5. Therefore, by Lagrange's theorem A_5 may be a candidate to exists within PSp (4, 2) as a subgroup.

In order to find the possibility of the existence of A_5 within PSp(4, 2), we need some additional information of PSp (4, 2). Since A_5 is obtained by $a^2 = 2$, $b^3 = 1$ and $(ab)^5 = 1$.

$\Delta(2,3,5) = < a, b; a^2 = b^3 = (ab)^5 = 1 >$

By [17], β and γ are representations of conjugacy classes C_β and C_γ of the respective elements. Let $|\beta| = 2$ and $|\gamma| = 3$ such that $\beta\gamma = \alpha$, where $|\alpha| = 5$ in PSp (4, 2). To show that A_5 is a subgroup of PSp (4, 2), we need to use character (table 1.1). There are 3 conjugacy classes of order 2, 1 conjugacy class of order 5 and 2 conjugacy classes of order 3. To find conjugacy classes, we use the basic formula. We see δ_{04} involved but δ_{02} and δ_{03} are not involved in the relation. The general formula for calculating the conjugacy classes be

$$\#<\beta,\gamma> = \frac{|G|}{|C_G(\beta)||C_G(\gamma)|}\sum_{i=1}^{11}\frac{\delta_i(\beta)\delta_i(\gamma)\overline{\delta_i(\alpha)}}{\delta_i(1)}$$

where solutions of equations $\beta.\gamma = \alpha$ are obtained by $\#<\beta.\gamma>$, the degree of characters of G denoted by $\chi_i(1)$, $\chi_i(\beta)$ and characters values of ith character denoted by $\chi_i(\gamma)$ and $\overline{\delta(\alpha)}$, Where $\overline{\delta_1(\alpha)}$ be the conjugate of α. Note that $a \in \delta_{04}$, $b \in \delta_{07}$ and $(ab) \in \delta_{11}$ then,

$\# < 2.3 = 5 > = 720 / 16.18 [1 + 0 + 0 + \ldots 0] = 5$

Clearly, there exist five triangular relations in PSp (4, 2) and form alternating group A_5 within PSp (4, 2). Now the only question is left to be answered here is that how many of such A_5 are conjugate? For this, we see that

$|C_G(\alpha)| = 5$, where $|\alpha| = 5$. Hence the class β_{04} of PSp (4,2) and β_{07} is contained in (2, 3, 5) triangular relation. Let $a \in \delta_{04}$, $b \in \delta_{09}$ and $(ab) \in \delta_{11}$,
$\# < 2.3$
$= 5 > = 720/16.18 [1 + 0 + 0 + \ldots 0]$
$= 5$

Hence δ_{04} of PSp (4, 2) and δ_{07} also involved in the above relation. Hence for each $\#< 2.3 = 5 > = 5$ relations, we have
$< a, b; a^2 = b^3 = (ab)^5 = 1 > = A_5$ & $|C_G(5)| = 5$.

Clearly, A_5 exist in the relation (2,3, 5) by different conjugate values of the elements *a* and *b*. If |a| and |b|, then by [23] , we have seen from Table 1.1 PSp (4,2) (see below) contained in *PSp* (4, 2), then order of $ab = 5$. In order to find the number of conjugate subgroups with A_5, we have $|\alpha| = 5$ and $|\beta| = 2$, $|\gamma| = 3$ in *PSp* (4, 2) and $|\alpha| = |\beta.\gamma| = 5$
then $\alpha \in C_G(5)$. $x(\beta.\gamma)\ x^{-1} = x$
$\alpha x^{-1} = \alpha$

This implies that

$x\beta x^{-1} x\gamma x^{-1} = \alpha$
so $(x\beta x^{-1})(x\gamma x^{-1}) = \alpha$

The same α is obtained by conjugating β and γ with $\alpha = 5$ in $C_G(\alpha)$. So the total number of pairs of β and γ is equal in number to order of the centralizer of α.
Thus, all the 5 relations become conjugate by conjugating β and γ with the centralizer of α. This concludes that there are just two non-conjugate classes of simple group $\Delta(2, 3, 5) \cong A_5$ with in *PSp* (4, 2) as required.

2.5. Theorem: Let G be a Projective S ymplectic group PSp (4, 3). Then there exist two non-conjugate classes of simple group isomorphic to A_5 involved by classes C_β, C_γ and C_α such that $\beta^2 = \gamma^3 = 1 = (\beta.\gamma) = \alpha^5$ in PSp (4, 3).

Proof: Let $G_1 \leq$ PSp (4, 3) be a subgroup isomorphic to A_5. Therefore, by Lagrange's theorem A_5 may be a candidate to exists within PSp (4, 3) as a subgroup.

In order to find the possibility of the existence of A_5 within PSp(4, 3), we need some additional information of PSp (4, 3). Since A_5 is obtained by $a^2 = 2$, $b^3 = 1$ and $(ab)^5 = 1$.

Class	δ_{01}	δ_{02}	δ_{03}	δ_{04}	δ_{05}	δ_{06}	δ_{07}	δ_{08}	δ_{09}	δ_{10}	δ_{11}
n(l)	1	15	15	45	90	90	40	120	40	120	144
Order	1	[2]	[2,2]	[2,2,2]	[4]	[4,4]	[3]	[6]	[3,3]	[6]	[5]
χ_{01}	1	1	1	1	1	1	1	1	1	1	1
χ_{02}	1	-1	-1	1	1	-1	1	-1	1	-1	1
χ_{03}	5	-3	1	1	-1	-1	2	0	-1	1	0
χ_{04}	5	3	-1	1	-1	1	2	0	-1	-1	0
χ_{05}	5	-1	3	1	-1	1	-1	-1	2	0	0
χ_{06}	5	1	-3	1	-1	-1	-1	1	2	0	0
χ_{07}	9	-3	-3	0	1	1	1	0	0	0	-1
χ_{08}	9	3	3	1	1	-1	0	0	0	0	-1
χ_{09}	10	-2	2	-2	0	0	1	1	1	-1	0
χ_{10}	10	-2	2	-2	0	0	1	1	1	-1	0
χ_{11}	16	0	0	0	0	0	-2	0	-2	0	1

Table (1.1)

By [17], β and γ are representations of conjugacy classes C_β and C_γ of the respective elements. Let $|\beta| = 2$ and $|\gamma| = 3$ such that $\beta\gamma = \alpha$, where $|\alpha| = 5$ in PSp (4, 3). To show that A_5 is a subgroup of PSp (4, 3), we need to use character (Table 1.2). Clearly there are 2 classes of elements of order 2.
So first or second or both involved in the relation. Since only even permutations involved in the construction of A_5, so δ_{02} is not involved in the relation as it is odd order class. Only δ_{03} involved in the relation (2,3,5). Similarly, 4 classes of elements of order 3.

There is only one class of the elements of order 5. The general formula for calculating the conjugacy classes in the construction of triangular group is
$\# <\beta, \gamma >=$

$$\frac{|G|}{|C_G(\beta)||C_G(\gamma)|}\sum_{i=1}^{20}\frac{\delta_i(\beta)\delta_i(\gamma)\overline{\delta_\iota(\alpha)}}{\delta(1)}$$

where solutions of equations $\beta.\gamma = \alpha$ are obtained by $\# < \beta.\gamma >$, the degree of characters of G denoted by $\chi_i(1)$, $\chi_i(\beta)$ & characters values of ith character denoted by

$\chi_i(\gamma)$ and $\overline{\delta_\iota(\alpha)}$

Where $\overline{\delta_\iota(\alpha)}$ be the conjugate of α.

Note that $a \in \delta_{03}$, $b \in \delta_{06}$ and $(ab) \in \delta_{10}$ then,

$\#<2.3 = 5 >= 2^6.3^4.5/2^7.3^4[2] = 5$

Clearly, there exist 5 triangular relations in PSp (4, 3) and form alternating group
A_5 within PSp (4, 3). Now the only question is left to be answered here is that how many of such A_5 are conjugate?
For this, we see that
$|C_G(\alpha)| = 5$,
where $|\alpha| = 5$.

Hence the class δ_{03} of PSp (4, 3) and δ_{06} is in the construction of (2, 3, 5) relation.
Now, we observe that $a \in \delta_{03}$, $b \in \delta_{07}$ and $(ab) \in \delta_{10}$,
$\# < 2.3 = 5 > = 2^6.3^4.5 / 2^7.3^4 [2] = 5$.

Hence the class δ_{03} of PSp (4, 3) and δ_{07} also involved in the above relation. Thus, for each of the relation

$\# < 2.3 = 5 > = 5$,

we have
$A_5 = < a, b; a^2 = b^3 = (ab)^5 = 1 >$ Since $\# < \beta . \gamma = \alpha > = 5$,

we see
$|C_G(5)| = 5$

Clearly, A_5 exist in the relation (2, 3, 5) by different conjugate values of the elements a and b. If |a| and |b| contained in PSp (4, 3), then order of ab = 5. In order to find the number of conjugate subgroups with A_5,

we have
$|\alpha| = 5$ and $|\beta| = 2$, $|\gamma| = 3$ in PSp (4, 3) and $|\alpha| = |\beta.\gamma| = 5$.
then
$\alpha \in C_G(5). x(\beta.\gamma)x^{-1} = x$
$\alpha x^{-1} = \alpha,$
this implies that $x\beta x^{-1} x\gamma \; x^{-1} = \alpha$
thus
$(x\beta x^{-1})(x\gamma x^{-1}) = \alpha$
The same α is obtained by conjugating β and γ with $\alpha = 5$ in $C_G(\alpha)$.
So the total number of pairs of β and γ is equal in number to order of the centralizer of α.

Thus, all the 5 relations become conjugate by onjugating β and γ with the centralizer of α. This concludes that there are just two non-conjugate classes of simple group $\Delta(2, 3, 5) \cong A_5$
Within PSp (4,3) as required.
The character table used in this theorem is given below in Table (2.1).

Class	δ_{01}	δ_{02}	δ_{03}	δ_{04}	δ_{05}	δ_{06}	δ_{07}	δ_{08}	δ_{09}	δ_{10}
n(l)	1	45	270	40	40	240	480	540	3240	5184
Order	1	[2]	[2]	[3]	[3]	[3]	[3]	[4]	[4]	[5]
χ_{01}	81	9	-3	0	0	0	0	-3	-1	1
χ_{02}	64	0	0	-8	-8	4	-2	0	0	-1
χ_{03}	60	-4	4	6	6	-3	-3	0	0	0
χ_{04}	45	-3	-3	/T	T	0	0	1	1	0
χ_{05}	45	-3	-3	T	/T	0	0	1	1	0
χ_{06}	40	-8	0	/S	S	-2	1	0	0	0
χ_{07}	40	-8	0	S	/S	-2	1	0	0	0
χ_{08}	30	6	2	/R	R	-3	0	2	0	0
χ_{09}	30	6	2	R	/R	-3	0	2	0	0
χ_{10}	30	-10	2	3	3	3	3	-2	0	0
χ_{11}	24	8	0	6	6	0	3	0	0	-1
χ_{12}	20	4	4	2	2	5	-1	0	0	0
χ_{13}	15	7	3	-3	-3	0	3	-1	1	0
χ_{14}	15	-1	-1	6	6	3	0	3	-1	0
χ_{15}	10	2	-2	/Q	Q	1	1	2	0	0
χ_{16}	10	2	-2	Q	/Q	1	1	2	0	0
χ_{17}	6	-2	2	-3	-3	3	0	2	0	1
χ_{18}	5	-3	1	/P	P	-1	2	1	-1	0
χ_{19}	5	-3	1	P	/P	-1	2	1	-1	0
χ_{20}	1	1	1	1	1	1	1	1	1	1

Class	δ_{11}	δ_{12}	δ_{13}	δ_{14}	δ_{15}	δ_{16}	δ_{17}	δ_{18}	δ_{19}	δ_{20}
n(l)	360	360	720	720	1440	2160	2880	2880	2160	2160
Order	[6]	[6]	[6]	[6]	[6]	[6]	[9]	[9]	[12]	[12]
χ_{01}	0	0	0	0	0	0	0	0	0	0
χ_{02}	0	0	0	0	0	0	1	1	0	0
χ_{03}	2	2	-1	-1	-1	1	0	0	0	0
χ_{04}	/W	W	0	0	0	0	0	0	-/Y	-Y
χ_{05}	W	/W	0	0	0	0	0	0	-Y	-/Y
χ_{06}	/V	V	/V	V	1	0	-Y	-/Y	0	0
χ_{07}	V	/V	V	/V	1	0	-/Y	-Y	0	0
χ_{08}	U	/U	X	-X	0	-1	0	0	Y	/Y
χ_{09}	/U	U	-X	X	0	-1	0	0	/J	J
χ_{10}	-1	-1	-1	-1	-1	-1	0	0	1	1
χ_{11}	2	2	2	2	-1	0	0	0	0	0
χ_{12}	-2	-2	1	1	1	1	-1	-1	0	0
χ_{13}	1	1	-2	-2	1	0	0	0	-1	-1
χ_{14}	2	2	-1	-1	2	-1	0	0	0	0
χ_{15}	/P	P	-1	-1	-1	1	-Y	-/Y	/J	J
χ_{16}	P	/P	-1	-1	-1	1	-/Y	-Y	J	/J
χ_{17}	1	1	1	1	-2	-1	0	0	-1	-1
χ_{18}	/U	U	-I	I	0	1	/Y	Y	-/Y	-Y
χ_{19}	U	/U	I	-I	0	1	Y	/Y	-Y	-/Y
χ_{20}	1	1	1	1	1	1	1	1	1	1

(Table 1.2)

P=$(1+3\sqrt{-3})/2$, /P=$1/\{(1+3\sqrt{-3})/2\}$,

Q=$(-7+3\sqrt{-3})/2$,/Q=$1/\{(1+3\sqrt{-3})/2\}$

R=$(-3+9\sqrt{-3})/2$,/R=$1/\{(-3+9\sqrt{-3}/2\}$,

S=$-5-3\sqrt{-3})$, /S=$1/\{-5-3\sqrt{-3})\}$,

T=$(9-9\sqrt{-3})/2$, /T=$1/\{(9-9\sqrt{-3})/2\}$,

U=$(-3-\sqrt{-3})/2$, /U=$1/\{(-3-\sqrt{-3})/2\}$

V=$1-\sqrt{-3}$, /V=$1/\{1-\sqrt{-3}\}$,

W=$(-3+3\sqrt{-3})/2$, /W=$1/\{(-3+3\sqrt{-3})/2\}$,

X=$\sqrt{-3}$, /X=$1/\sqrt{-3}$, Y=$(1-\sqrt{-3})/2$,

Y=$1/\{(1-\sqrt{-3})/2\}$

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Authorship and Contribution Declaration

Sr.#	**Author-s Full Name**	**Contribution to Paper**	**Signature**
1	Dr.Abid Mahboob (Main/principal Author)	Theme and idea of topic which includes representation theory & methodology developed regarding	
2	Dr. Muhammad Altaf (2nd Author)	Involved in collection of data & working software used to collect character Table	Signature by the Corresponding author on Behalf of Co-Authors
3	Mr. Sajid Mahboob (3rd Author)	Literature review of group theory	
4	Mr. Taswer Hussain	Contribution in theory of projective Symplectic group	