Technical Journal, University of Engineering and Technology (UET) Taxila, Pakistan *Vol. 27 No. 1-2022*
ISSN:1813-1786 (Print) 2313-7770 (Online)

A Novel Generalized Hybrid Aggregation Operators Based on Picture Hesitant Fuzzy Sets and Their Applications in Decision Making

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Abstract- To evaluate the beneficial and valuable decision from the collection of preferences, the MADM "multi-attribute decision-making" technique is more effective for evaluating our main problem. Moreover, the main theory of picture hesitant fuzzy (PHF) information is the modified and generalized version of the fuzzy, intuitionistic, picture, hesitant fuzzy information and some other extensions, and due to these reasons, they receive a lot of attention from different scholars. The major influence of this theory is to pioneer the theory of generalized hybrid aggregation operators under the consideration of PHF information, called PHF generalized weighted averaging (PHFGWA), PHF generalized ordered weighted averaging (PHFGOWA), PHF generalized hybrid weighted averaging (PHFGHWA) operators, evaluated their valuable and flexible properties with some dominant results. Further, we know that without a decision-making technique it is very awkward to find our required information, therefore, in this manuscript, we also evaluated a MADM procedure under the availability of diagnosed operators to show the supremacy and worth of the pioneered information. Finally, we aim to evaluate the comparative analysis of the pioneered information with some prevailing operators to enhance the capability and accuracy of the proposed information.

Keywords- Picture hesitant fuzzy sets; Correlation coefficient; Mean; Variance; Decision making.

I. INTRODUCTION

Under the consideration of classical information, it is very complicated to find the valuable and dominant preference from the family of preferences. Because in classical information we have only two possibilities like "0" and "1" and for this reason, we lose a lot of valuable information. For evaluating such an awkward type of situation, the major information of fuzzy set (FS) [1], computed with a shape whose main value belongs to the unit interval. Certain applications have been done by many students, for instance, in medical diagnosis [2-3], pattern recognition [4-5], and decision making [6-7]. FS has a lot of applications but due to its weak shape, certain people try to avoid theme to use it. Because if someone talked about non-membership, then the theory of FS has been unsuitable. Therefore, how to evaluate the above problem, intuitionistic FS (IFS) [8] was computed with a shape whose main values (like truth and falsity grades) belong to the unit interval. Certain applications have been done by many students, for instance, in [9-10] pioneered the theory of partial functional differential equations using IFS and evaluated their existence and uniqueness and evaluated the fourth order Runge-Kutta gill method in the presence of IFSs. Moreover, in [11] diagnosed the IFS by using their shape in the form of complex numbers. in [12] formulated the theory of IFS using the shape of the graph. Certain more applications have been done by many students, for instance, in decision-making [13-15] problems.

By facing information in the form of yes, abstinence, and no, certain scholars have decided to compute a structure that fulfills evaluate the above dilemma, for instance, if someone cast his vote in the election, some people cast their vote in the support of someone, someone against, and someone abstinence during casting of voting in the election. The theory of IFS has not suitability utilized or managed with the above scenario. For this, the main and valuable theory of IFS was modified and pioneered the theory of picture FS (PFS) [16], where truth, abstinence, and falsity grades are the parts of the PFS with a condition that is the total of the triple will be contained in the unit interval. Certain more applications have been done by many students, for instance, in TOPSIS methods [17], aggregation operators [18], Muirhead mean operators [19], and decision-making [21-23] problems were evaluated based on PFSs.

To provide information in the shape of the group against each element belonging to a universal set, the theory of FSs, IFSs and PFSs has failed for managing such type of information. Because they can deal only with one dimension of information in a singleton set. Therefore, the major information on hesitant FS (HFS) [24] by enhancing the main theory of FSs. HFS is computed with a shape whose main value belongs to the unit interval. Certain applications have been done by many students, for instance, aggregation information [25], measures [26], and certain different fields [27-35]. Furthermore, by facing the information of yes, abstinence, and no in the shape of the collection of groups of information, certain scholars have decided to compute a structure that fulfills evaluate the above dilemma, for instance, if the group of people cast his vote in the election, some the group of people casts their vote in the support of someone, the group of people against, and the group of people abstinence during casting of voting in the election. The theory of FSs, IFSs, and PFSs has not been suitability utilized or managed with the above scenario. For this, the main and valuable theory of IFS was modified and diagnosed the theory of picture HFS (PHFS) [36], where truth, abstinence, and falsity grades are the parts of the PHFS in the shape of the group of information with a condition that is the total of the maximum of the triplet will be contained in the unit interval. Certain more applications have been done by many students, for instance, in [37] established the aggregation operators based on PHFSs. In [38] established the generalized similarity measures based on PHFSs. In [39] established the similarity measures based on PHFSs. Further, In [40] diagnosed the correct and perfect version of PFSs. In [41] also described the theory of another view of picture fuzzy soft sets. It is a very challenging task for authors how to aggregate the collection of finite numbers of information into one set, for this the aggregation operator is massively beneficial because the aggregation operator was computed based on t-norm and t-conorm. But in some situations, these aggregation operators are not working feasibly, because in the presence of Bonferroni mean operators, Maclaurin operators, and Hamacher operators, why people used the old and so-called aggregation operators because the simple aggregation operator is the special case of the Bonferroni mean operators, Maclaurin operators, and Hamacher operators by using some value of parameters. From the information explained above, we clear that the theory of PHFS is massively valuable and feasible for evaluating awkward and unreliable information in real-life situations, based on their advantages, the main theme of this analysis is described below:

1. To pioneer the theory of generalized hybrid aggregation operators under the consideration of PHF information, called PHFGWA, PHFGOWA, and PHFGHWA operators.
2. To evaluate their valuable and flexible properties with some dominant results.
3. To discuss a MADM procedure under the availability of diagnosed operators to show the supremacy and worth of the pioneered information.
4. To evaluate the comparative analysis of the pioneered information with some prevailing operators to enhance the capability and accuracy of the proposed information.

The main construction of this theory is in the shape: Section 2 reviewed the old idea of PFSs, HFSs, PHFSs, aggregation operators, and their properties. In section 3, we pioneered the theory of generalized hybrid aggregation operators under the consideration of PHF information, called PHFGWA, PHFGOWA, and PHFGHWA operators. In section 4, we discussed a MADM procedure under the availability of diagnosed operators to show the supremacy and worth of the pioneered information. Further, we evaluated the comparative analysis of the pioneered information with some prevailing operators to enhance the capability and accuracy of the proposed information. Concluding information is available in section 5.

II. PRELIMINARIES

In this study, we review some basic drawbacks like PFS, HFS, PHFS, and their properties. The drawback of the existing Distance measure (DM), similarity measure (SM), Cardinal number, normalized generalized DM, normalized Euclidean DM, and normalized hamming DM, are also described. Throughout this article represents the grade of positive, abstinence, and negative, which is denoted by the fixed set.

Definition 1: [16] A PFS is of the form $\Gamma = \{(m(\mathcal{v}), a(\mathcal{v}), n(\mathcal{v})) : \mathcal{v} \in X\}$, with $0 \leq m(\mathcal{v}) + a(\mathcal{v}) + n(\mathcal{v}) \leq 1$, with $m, a, n: X \rightarrow [0,1]$. Further, $(m(\mathcal{v}), a(\mathcal{v}), n(\mathcal{v}))$ is called picture fuzzy number (PFN) and $\pi(\mathcal{v}) = 1 - (m(\mathcal{v}) + a(\mathcal{v}) + n(\mathcal{v}))$ is called refusal grade.

Definition 2: [24] A HFS H on X is stated by: $H = \{< \mathcal{v}, m(\mathcal{v}) >: m(\mathcal{v})$ is a finite subset of $[0,1] \,\forall\, \mathcal{v} \in X\}$. Furthermore, $m(\mathcal{v})$ is called a hesitant fuzzy number (HFN).

Definition 3: [24] Let m, m_1 and m_2 be three HFE with $\partial > 0$. Then

1. $m_1 \oplus m_2 = \coprod_{t_1 \in m_1, t_2 \in m_2} \{t_1 + t_2 - t_1 t_2\}$
2. $m_1 \otimes m_2 = \coprod_{t_1 \in m_1, t_2 \in m_2} \{t_1 t_2\}$
3. $m^{\partial} = \coprod_{t \in m} \{t^{\partial}\}$;
4. $\partial m = \coprod_{t \in m} \{1 - (1 - t)^{\partial}\}$

Definition 4: [36] A PHFS is of the form $\Gamma = \{(m_{ŧ}(\mathcal{v}), a_{ŧ}(\mathcal{v}), n_{ŧ}(\mathcal{v})): \mathcal{v} \in X\}, ŧ = 1,2,3,\ldots,z$, with $0 \leq \max(m_{ŧ}(\mathcal{v})) + \max(a_{ŧ}(\mathcal{v})) + \max(n_{ŧ}(\mathcal{v})) \leq 1$, with $m_{ŧ}, a_{ŧ}, n_{ŧ}$ is a finite subsets of [0,1]. Further, $(m(\mathcal{v}), a(\mathcal{v}), n(\mathcal{v}))$ is called PHF number (PHFN) and $\pi_{ŧ}(\mathcal{v}) = 1 - (m_{ŧ}(\mathcal{v}) + a_{ŧ}(\mathcal{v}) + n_{ŧ}(\mathcal{v}))$ is called refusal grade.

Definition 5: [16] The generalized weighted averaging (GWA) operator is given by: GWA: $\Omega^{\Xi} \to \Omega$ by

$$\mathrm{GWA}(\Xi_1, \Xi_2, \ldots, \Xi_{\Xi}) = \left(\sum_{ŧ=1}^{\Xi} \Sigma_{ŧ} \Xi_{ŧ}^{\partial}\right)^{\frac{1}{\partial}}$$

where $\partial > 0$ and the terms $\Sigma = (\Sigma_1, \Sigma_2, \ldots, \Sigma_{\Xi})^{\mathrm{T}}, \Sigma_{ŧ} \in [0,1]$ where $\sum_{ŧ=1}^{\Xi} \Sigma_{ŧ} = 1$, represents the weight vector.

Definition 6: [16] The generalized ordered weighted averaging (GWA) operator is given by: GOWA: $\Omega^{\Xi} \to \Omega$ by

$$\mathrm{GOWA}(\Xi_1, \Xi_2, \ldots, \Xi_{\Xi}) = \left(\sum_{ŧ=1}^{\Xi} \Sigma_{ŧ} {\Xi'}_{o(ŧ)}^{\partial}\right)^{\frac{1}{\partial}}$$

where $\partial > 0$ and the terms $\Sigma = (\Sigma_1, \Sigma_2, \ldots, \Sigma_{\Xi})^{\mathrm{T}}, \Sigma_{ŧ} \in [0,1]$ where $\sum_{ŧ=1}^{\Xi} \Sigma_{ŧ} = 1$, represents weight vector with permutations $\Xi'_{o(ŧ)} \leq \Xi'_{o(ŧ-1)}$.

Definition 7: [16] The generalized hybrid weighted averaging (GHWA) operator is given by:
GHWA: $\Omega^{\Xi} \to \Omega$ by

$$\mathrm{GHWA}(\Xi_1, \Xi_2, \ldots, \Xi_{\Xi}) = \left(\sum_{ŧ=1}^{\Xi} \Sigma_{ŧ} {\Xi'}_{o(ŧ)}^{\partial}\right)^{\frac{1}{\partial}}$$

where $\partial > 0$ and the terms $\Sigma = (\Sigma_1, \Sigma_2, \ldots, \Sigma_{\Xi})^{\mathrm{T}}, \Sigma_{ŧ} \in [0,1]$ where $\sum_{ŧ=1}^{\Xi} \Sigma_{ŧ} = 1$, represents weight vector with permutations $\Xi'_{o(ŧ)} \leq \Xi'_{o(ŧ-1)}$ where $\Xi'_{ŧ} = \Xi \acute{\omega}_{ŧ} \Xi_{ŧ}$.

III. Some Generalized Hybrid Aggregation Operators Based on PHFSs

To evaluate the beneficial and valuable decision from the collection of preferences, the MADM technique is more effective for evaluating our main problem. Moreover, the main theory of PHF information is the modified and generalized version of the fuzzy, intuitionistic, picture, hesitant fuzzy information and some other extensions, and due to these reasons, they receive a lot of attention from different scholars. The major influence of this theory is to pioneer the theory of generalized hybrid aggregation operators under the consideration of PHF information, called PHFGWA, PHFGOWA, and PHFGHWA operators, evaluated their valuable and flexible properties with some dominant results.

Definition 8: Let $\Gamma = \{(m_{pŧ}(\mathcal{v}), a_{pŧ}(\mathcal{v}), n_{pŧ}(\mathcal{v})): \mathcal{v} \in X\}$ and $\mathcal{Q} = \{(m_{qŧ}(\mathcal{v}), a_{qŧ}(\mathcal{v}), n_{qŧ}(\mathcal{v})): \mathcal{v} \in X\}$ be two PHFNs with $\partial > 0$. Then

1. $\Gamma \oplus \mathcal{Q} =$
$$\coprod_{\substack{t_1' \in m_{pŧ}, a_1' \in a_{pŧ}, f_1' \in n_{pŧ}, \\ t_2' \in m_{qŧ}, a_2' \in a_{qŧ}, f_2' \in n_{qŧ}}} \left(\begin{pmatrix} t_1' + t_2' - \\ t_1' t_2' \end{pmatrix}, \\ (a_1' a_2'), (f_1' f_2')\right);$$
2. $\Gamma \otimes \mathcal{Q} =$
$$\coprod_{\substack{t_1' \in m_{pŧ}, a_1' \in a_{pŧ}, f_1' \in n_{pŧ}, \\ t_2' \in m_{qŧ}, a_2' \in a_{qŧ}, f_2' \in n_{qŧ}}} \left((t_1' t_2'), \begin{pmatrix} a_1' + a_2' - \\ a_1' a_2' \end{pmatrix}, \\ \begin{pmatrix} f_1' + f_2' - \\ f_1' f_2' \end{pmatrix}\right)$$
;
3. $\partial \Gamma = \coprod_{t_1' \in m_{pŧ}, a_1' \in a_{pŧ}, f_1' \in n_{pŧ}} \left((1 - (1 - t_1')^{\partial}), {a_1'}^{\partial}, {f_1'}^{\partial}\right);$
4. $\Gamma^{\partial} = \coprod_{t_1' \in m_{pŧ}, a_1' \in a_{pŧ}, f_1' \in n_{pŧ}} \left({t_1'}^{\partial}, (1 - (1 - a_1')^{\partial}), (1 - (1 - f_1')^{\partial})\right).$

Definition 9: Let $\Gamma = \{(m_{pŧ}(\mathcal{v}), a_{pŧ}(\mathcal{v}), n_{pŧ}(\mathcal{v})): \mathcal{v} \in X\}$ be PHFN. Then

$$Ş(\Gamma) = \frac{1}{3}\left\{\left(\frac{1}{\alpha}\sum_{ŧ=1}^{\alpha} m_{pŧ} + \frac{1}{\beta}\sum_{ŧ=1}^{\beta} a_{pŧ} + \frac{1}{\gamma}\sum_{ŧ=1}^{\gamma} n_{pŧ}\right)\right\}$$

Is called the score function and the accuracy function is denoted and defined by:

$$H(\Gamma) = \frac{1}{3}\left\{\left(\frac{1}{\alpha}\sum_{ŧ=1}^{\alpha} m_{pŧ} - \frac{1}{\beta}\sum_{ŧ=1}^{\beta} a_{pŧ} - \frac{1}{\gamma}\sum_{ŧ=1}^{\gamma} n_{pŧ}\right)\right\}$$

Where α, β, γ represents the length of the positive, abstinence, and negative subsets of [0,1]. If we considered the two PHFNs $\Gamma = \{(m_{pŧ}(\mathcal{v}), a_{pŧ}(\mathcal{v}), n_{pŧ}(\mathcal{v})): \mathcal{v} \in X\}$ and $\mathcal{Q} = \{(m_{qŧ}(\mathcal{v}), a_{qŧ}(\mathcal{v}), n_{qŧ}(\mathcal{v})): \mathcal{v} \in X\}$, then When $Ş(\Gamma) > Ş(\mathcal{Q}) \Rightarrow \Gamma > \mathcal{Q}$; When $Ş(\Gamma) < Ş(\mathcal{Q}) \Rightarrow \Gamma < \mathcal{Q}$; When $Ş(\Gamma) = Ş(\mathcal{Q}) \Rightarrow$ When $H(\Gamma) > H(\mathcal{Q}) \Rightarrow \Gamma > \mathcal{Q}$; When $H(\Gamma) < H(\mathcal{Q}) \Rightarrow \Gamma < \mathcal{Q}$; When $H(\Gamma) = H(\mathcal{Q}) \Rightarrow \Gamma = \mathcal{Q}$.

Example 1: Let $\Gamma = \begin{pmatrix} (0.3,0.1), (0.4,0.3,0.2), \\ (0.4,0.2,0.1,0.1) \end{pmatrix}$ and $\mathcal{Q} = \begin{pmatrix} (0.31,0.21,0.11), (0.31,0.21), \\ (0.51,0.11) \end{pmatrix}$ be two PHFNs with $\partial = 2$. Then

$$Ş(\Gamma) = \frac{1}{3}\left\{\left(\frac{1}{\alpha}\sum_{ŧ=1}^{\alpha} m_{pŧ} + \frac{1}{\beta}\sum_{ŧ=1}^{\beta} a_{pŧ} + \frac{1}{\gamma}\sum_{ŧ=1}^{\gamma} n_{pŧ}\right)\right\}$$
$$= \frac{1}{3}\left\{\left(\frac{1}{2}(0.3 + 0.1) + \frac{1}{3}(0.4 + 0.3 + 0.2) + \frac{1}{4}(0.4 + 0.2 + 0.2 + 0.1)\right)\right\}$$

$$= \frac{1}{3}(0.2 + 0.3 + 0.225) = 0.24167$$

$$Ş(Q) = \frac{1}{3}\left\{\left(\frac{1}{\alpha}\sum_{ŧ=1}^{\alpha} m_{pŧ} + \frac{1}{\beta}\sum_{ŧ=1}^{\beta} a_{pŧ} + \frac{1}{\gamma}\sum_{ŧ=1}^{\gamma} n_{pŧ}\right)\right\}$$

$$= \frac{1}{3}\left\{\left(\frac{1}{3}(0.31 + 0.21 + 0.11) + \frac{1}{2}(0.31 + 0.21) + \frac{1}{2}(0.51 + 0.11)\right)\right\}$$

$$= \frac{1}{3}(0.21 + 0.26 + 0.31) = 0.26$$

$Ş(\Gamma) < Ş(Q)$, then $\Gamma < Q$. Similarly, if $Ş(\Gamma) > Ş(Q)$. If $Ş(\Gamma) = Ş(Q)$, we use the accuracy function formula.

Definition 10: The PHFGWA operator is given by: PHFGWA: $\Omega^{\Xi} \to \Omega$ by

$$\text{PHFGWA}(\Gamma_1, \Gamma_2, .., \Gamma_{\Xi}) = \left(\sum_{ŧ=1}^{\Xi} \Sigma_ŧ \Gamma_ŧ^{\partial}\right)^{\frac{1}{\partial}}$$

where Ω has represented the family of all PHFNs with $\partial > 0$. The PHFN is of the form $\Gamma = \{(m_{pŧ}(\mathcal{v}), a_{pŧ}(\mathcal{v}), n_{pŧ}(\mathcal{v})): \mathcal{v} \in X\}(ŧ = 1,2, .., \Xi)$.

Theorem 1: Let $\Gamma = \{(m_{pŧ}(\mathcal{v}), a_{pŧ}(\mathcal{v}), n_{pŧ}(\mathcal{v})): \mathcal{v} \in X\}(ŧ = 1,2, .., \Xi)$ be the family of PHFNs with $\partial > 0$. Then by using the theory of PHFGWA, we again received PHFN, such that

$$\text{PHFGWA}(\Gamma_1, \Gamma_2, .., \Gamma_{\Xi}) = \left(\begin{array}{c} \left(1 - \prod_{ŧ=1}^{\Xi} \coprod_{t_ŧ' \in m_{pŧ}} \left(1 - (t_ŧ')^{\partial}\right)^{\Sigma_ŧ}\right)^{\frac{1}{\partial}}, \\ \left(1 - \left(1 - \prod_{ŧ=1}^{\Xi} \coprod_{a_ŧ' \in a_{pŧ}} \left(1 - (1 - a_ŧ')^{\partial}\right)^{\Sigma_ŧ}\right)^{\frac{1}{\partial}}\right), \\ \left(1 - \left(1 - \prod_{ŧ=1}^{\Xi} \coprod_{f_ŧ' \in n_{pŧ}} \left(1 - (1 - f_ŧ')^{\partial}\right)^{\Sigma_ŧ}\right)^{\frac{1}{\partial}}\right) \end{array}\right)$$

Proof:

1. First, we have proven that

$$\sum_{ŧ=1}^{\Xi} \Sigma_ŧ \Gamma_ŧ^{\partial} = \left(\begin{array}{c} \left(1 - \prod_{ŧ=1}^{\Xi} \coprod_{t_ŧ' \in m_{pŧ}} \left(1 - (t_ŧ')^{\partial}\right)^{\Sigma_ŧ}\right), \\ \left(\prod_{ŧ=1}^{\Xi} \coprod_{a_ŧ' \in a_{pŧ}} \left(1 - (1 - a_ŧ')^{\partial}\right)^{\Sigma_ŧ}\right), \\ \left(\prod_{ŧ=1}^{\Xi} \coprod_{f_ŧ' \in n_{pŧ}} \left(1 - (1 - f_ŧ')^{\partial}\right)^{\Sigma_ŧ}\right) \end{array}\right)$$

Using mathematical induction, we prove the above information, such that

Case 1. If we considered $\Xi = 1$

$$\Gamma_1^{\partial} = \coprod_{t_1' \in m_{pŧ}, a_1' \in a_{pŧ}, f_1' \in n_{pŧ}} \left((t_1')^{\partial}, \left(1 - (1 - a_1')^{\partial}\right), \left(1 - (1 - f_1')^{\partial}\right)\right)$$

$$\Sigma_1 \Gamma_1^{\partial} = \coprod_{t_1' \in m_{pŧ}, a_1' \in a_{pŧ}, f_1' \in n_{pŧ}} \left(\left(1 - \left(1 - (t_1')^{\partial}\right)^{\Sigma_1}\right), \left(1 - (1 - a_1')^{\partial}\right)^{\Sigma_1}, \left(1 - (1 - f_1')^{\partial}\right)^{\Sigma_1}\right)$$

It is true for $\Xi = 1$.

Case 2. If, $\Xi = k$ we considered is right, then

$$\sum_{ŧ=1}^{k} \Sigma_ŧ \Gamma_ŧ^{\partial} = \left(\begin{array}{c} \left(1 - \prod_{ŧ=1}^{k} \coprod_{t_ŧ' \in m_{pŧ}} \left(1 - (t_ŧ')^{\partial}\right)^{\Sigma_ŧ}\right), \\ \left(\prod_{ŧ=1}^{k} \coprod_{a_ŧ' \in a_{pŧ}} \left(1 - (1 - a_ŧ')^{\partial}\right)^{\Sigma_ŧ}\right), \\ \left(\prod_{ŧ=1}^{k} \coprod_{f_ŧ' \in n_{pŧ}} \left(1 - (1 - f_ŧ')^{\partial}\right)^{\Sigma_ŧ}\right) \end{array}\right)$$

Then, we checked for $\Xi = k + 1$, and we get

$$\Sigma_{k+1} \Xi_{k+1}^{\partial} = \coprod_{\left(\begin{array}{c} t_{k+1}' \in m_{p(k+1)} \\ , a_{k+1}' \in a_{p(k+1)}, \\ f_{k+1}' \in n_{p(k+1)} \end{array}\right)} \left(\begin{array}{c} \left(1 - \left(1 - (t_{k+1}')^{\partial}\right)^{\Sigma_{k+1}}\right), \\ \left(1 - (1 - a_{k+1}')^{\partial}\right)^{\Sigma_{k+1}}, \\ \left(1 - (1 - f_{k+1}')^{\partial}\right)^{\Sigma_{k+1}} \end{array}\right)$$

and

$$\sum_{ŧ=1}^{k+1} \Sigma_ŧ \Gamma_ŧ^{\partial} = \left(\begin{array}{c} \left(\begin{array}{c} \left(1 - \prod_{ŧ=1}^{k} \coprod_{t_ŧ' \in m_{pŧ}} \left(1 - (t_ŧ')^{\partial}\right)^{\Sigma_ŧ}\right), \\ \left(\prod_{ŧ=1}^{k} \coprod_{a_ŧ' \in a_{pŧ}} \left(1 - (1 - a_ŧ')^{\partial}\right)^{\Sigma_ŧ}\right), \\ \left(\prod_{ŧ=1}^{k} \coprod_{f_ŧ' \in n_{pŧ}} \left(1 - (1 - f_ŧ')^{\partial}\right)^{\Sigma_ŧ}\right) \end{array}\right) \oplus \\ \coprod_{\left(\begin{array}{c} t_{k+1}' \in m_{p(k+1)} \\ , a_{k+1}' \in a_{p(k+1)}, \\ f_{k+1}' \in n_{p(k+1)} \end{array}\right)} \left(\begin{array}{c} \left(1 - \left(1 - (t_{k+1}')^{\partial}\right)^{\Sigma_{k+1}}\right), \\ \left(1 - (1 - a_{k+1}')^{\partial}\right)^{\Sigma_{k+1}}, \\ \left(1 - (1 - f_{k+1}')^{\partial}\right)^{\Sigma_{k+1}} \end{array}\right) \end{array}\right)$$

$$= \left(\begin{array}{c} \left(1 - \prod_{ŧ=1}^{k+1} \coprod_{t_ŧ{}' \in m_{pŧ}} \left(1 - (t_ŧ{}')^{\partial}\right)^{\Sigma_ŧ} \right), \\ \left(\prod_{ŧ=1}^{k+1} \coprod_{a_ŧ{}' \in a_{pŧ}} \left(1 - (1 - a_ŧ{}')^{\partial}\right)^{\Sigma_ŧ} \right), \\ \left(\prod_{ŧ=1}^{k+1} \coprod_{f_ŧ{}' \in n_{pŧ}} \left(1 - (1 - f_ŧ{}')^{\partial}\right)^{\Sigma_ŧ} \right) \end{array} \right)$$

For $\Xi = k + 1$, we get corrected.

$$\mathrm{PHFGWA}(\Gamma_1, \Gamma_2, .., \Gamma_\Xi) = \left(\sum_{ŧ=1}^{\Xi} \Sigma_ŧ \Gamma_ŧ^{\partial} \right)^{\frac{1}{\partial}}$$

$$= \left(\begin{array}{c} \left(1 - \prod_{ŧ=1}^{\Xi} \coprod_{t_ŧ{}' \in m_{pŧ}} \left(1 - (t_ŧ{}')^{\partial}\right)^{\Sigma_ŧ} \right)^{\frac{1}{\partial}}, \\ \left(1 - \left(1 - \prod_{ŧ=1}^{\Xi} \coprod_{a_ŧ{}' \in a_{pŧ}} \left(1 - (1 - a_ŧ{}')^{\partial}\right)^{\Sigma_ŧ} \right)^{\frac{1}{\partial}} \right), \\ \left(1 - \left(1 - \prod_{ŧ=1}^{\Xi} \coprod_{f_ŧ{}' \in n_{pŧ}} \left(1 - (1 - f_ŧ{}')^{\partial}\right)^{\Sigma_ŧ} \right)^{\frac{1}{\partial}} \right) \end{array} \right).$$

Next, some core properties of the initiated aggregation operators are investigated such as idempotency, monotonicity, and boundedness.

Theorem 2: Let $\Gamma_j = \{(m_{pŧj}, a_{pŧj}, n_{pŧj}) : \nu \in X\} (ŧ = 1,2, .., \Xi)$ be the family of PHFNs with $\partial > 0$. If $\Gamma = \Gamma_j$, then

$\mathrm{PHFGWA}(\Gamma_1, \Gamma_2, .., \Gamma_\Xi) = \Gamma$

Proof: If $\Gamma = \Gamma_j = \{(m_{pŧj}, a_{pŧj}, n_{pŧj}) : \nu \in X\}$, then

$$\mathrm{PHFGWA}(\Gamma_1, \Gamma_2, .., \Gamma_\Xi) = \left(\sum_{ŧ=1}^{\Xi} \Sigma_ŧ \Gamma_ŧ^{\partial} \right)^{\frac{1}{\partial}}$$

$$= \left(\sum_{ŧ=1}^{\Xi} \Sigma_ŧ \Gamma^{\partial} \right)^{\frac{1}{\partial}}$$

$$= \left(\begin{array}{c} \left(1 - \prod_{ŧ=1}^{\Xi} \coprod_{t_ŧ{}' \in m_{pŧ}} \left(1 - (t')^{\partial}\right)^{\Sigma_ŧ} \right)^{\frac{1}{\partial}}, \\ \left(1 - \left(1 - \prod_{ŧ=1}^{\Xi} \coprod_{a_ŧ{}' \in a_{pŧ}} \left(1 - (1 - a')^{\partial}\right)^{\Sigma_ŧ} \right)^{\frac{1}{\partial}} \right), \\ \left(1 - \left(1 - \prod_{ŧ=1}^{\Xi} \coprod_{f_ŧ{}' \in n_{pŧ}} \left(1 - (1 - f')^{\partial}\right)^{\Sigma_ŧ} \right)^{\frac{1}{\partial}} \right) \end{array} \right)$$

$$= \left(\begin{array}{c} \left(1 - \coprod_{t_ŧ{}' \in m_{pŧ}} \left(1 - (t')^{\partial}\right)^{\sum_{ŧ=1}^{\Xi} \Sigma_ŧ} \right)^{\frac{1}{\partial}}, \\ \left(1 - \left(1 - \coprod_{a_ŧ{}' \in a_{pŧ}} \left(1 - (1 - a')^{\partial}\right)^{\sum_{ŧ=1}^{\Xi} \Sigma_ŧ} \right)^{\frac{1}{\partial}} \right), \\ \left(1 - \left(1 - \coprod_{f_ŧ{}' \in n_{pŧ}} \left(1 - (1 - f')^{\partial}\right)^{\sum_{ŧ=1}^{\Xi} \Sigma_ŧ} \right)^{\frac{1}{\partial}} \right) \end{array} \right)$$

$$= \left(\begin{array}{c} \left(1 - \coprod_{t_ŧ{}' \in m_{pŧ}} \left(1 - (t')^{\partial}\right) \right)^{\frac{1}{\partial}}, \\ \left(1 - \left(1 - \coprod_{a_ŧ{}' \in a_{pŧ}} \left(1 - (1 - a')^{\partial}\right) \right)^{\frac{1}{\partial}} \right), \\ \left(1 - \left(1 - \coprod_{f_ŧ{}' \in n_{pŧ}} \left(1 - (1 - f')^{\partial}\right) \right)^{\frac{1}{\partial}} \right) \end{array} \right)$$

$$= \Gamma.$$

Theorem 3: Let $\Gamma_j = \{(m_{pŧj}, a_{pŧj}, n_{pŧj}) : \nu \in X\} (ŧ = 1,2, .., \Xi)$ be the family of PHFNs with $\partial > 0$. If $\Gamma_j \geq \Gamma_{j'}$, then

$\mathrm{PHFGWA}(\Gamma_1, \Gamma_2, .., \Gamma_\Xi) \geq \mathrm{PHFGWA}(\Gamma_1', \Gamma_2', .., \Gamma_\Xi')$

Proof: We considered $\Gamma_j \geq \Gamma_{ŧ'}$ it means that

$t_ŧ{}' \geq t_ŧ{}'', a_ŧ{}' \leq a_ŧ{}'', f_ŧ{}' \leq f_ŧ{}''$ for all ŧ, then

$$\left(1 - (t_ŧ{}')^{\partial}\right)^{\Sigma_ŧ} \leq \left(1 - \left(t_ŧ{}''\right)^{\partial}\right)^{\Sigma_ŧ}$$

$$\prod_{ŧ=1}^{\Xi}\coprod_{t_ŧ{}'\in m_{pŧ}}\left(1-(t_ŧ{}')^{\partial}\right)^{\Sigma_ŧ} \le \prod_{ŧ=1}^{\Xi}\coprod_{t_ŧ{}'\in m_{pŧ}}\left(1-\left(t_ŧ{}''\right)^{\partial}\right)^{\Sigma_ŧ}$$

$$\left(1-\prod_{ŧ=1}^{\Xi}\coprod_{t_ŧ{}'\in m_{pŧ}}\left(1-(t_ŧ{}')^{\partial}\right)^{\Sigma_ŧ}\right)^{\frac{1}{\partial}} \ge \left(1-\prod_{ŧ=1}^{\Xi}\coprod_{t_ŧ{}'\in m_{pŧ}}\left(1-\left(t_ŧ{}''\right)^{\partial}\right)^{\Sigma_ŧ}\right)^{\frac{1}{\partial}}$$

$$\left(1-\left(1-\prod_{ŧ=1}^{\Xi}\coprod_{a_ŧ{}'\in a_{pŧ}}\left(1-(1-a_ŧ{}')^{\partial}\right)^{\Sigma_ŧ}\right)^{\frac{1}{\partial}}\right) \le \left(1-\left(1-\prod_{ŧ=1}^{\Xi}\coprod_{a_ŧ{}'\in a_{pŧ}}\left(1-\left(1-a_ŧ{}''\right)^{\partial}\right)^{\Sigma_ŧ}\right)^{\frac{1}{\partial}}\right)$$

And

$$\left(1-\left(1-\prod_{ŧ=1}^{\Xi}\coprod_{f_ŧ{}'\in n_{pŧ}}\left(1-(1-f_ŧ{}')^{\partial}\right)^{\Sigma_ŧ}\right)^{\frac{1}{\partial}}\right) \le \left(1-\left(1-\prod_{ŧ=1}^{\Xi}\coprod_{f_ŧ{}'\in n_{pŧ}}\left(1-\left(1-f_ŧ{}''\right)^{\partial}\right)^{\Sigma_ŧ}\right)^{\frac{1}{\partial}}\right)$$

$$\begin{pmatrix} \left(1-\prod_{ŧ=1}^{\Xi}\coprod_{t_ŧ{}'\in m_{pŧ}}\left(1-(t_ŧ{}')^{\partial}\right)^{\Sigma_ŧ}\right)^{\frac{1}{\partial}}, \\ \left(1-\left(1-\prod_{ŧ=1}^{\Xi}\coprod_{a_ŧ{}'\in a_{pŧ}}\left(1-(1-a_ŧ{}')^{\partial}\right)^{\Sigma_ŧ}\right)^{\frac{1}{\partial}}\right), \\ \left(1-\left(1-\prod_{ŧ=1}^{\Xi}\coprod_{f_ŧ{}'\in n_{pŧ}}\left(1-(1-f_ŧ{}')^{\partial}\right)^{\Sigma_ŧ}\right)^{\frac{1}{\partial}}\right) \end{pmatrix} \ge \begin{pmatrix} \left(1-\prod_{ŧ=1}^{\Xi}\coprod_{t_ŧ{}'\in m_{pŧ}}\left(1-(t_ŧ{}'')^{\partial}\right)^{\Sigma_ŧ}\right)^{\frac{1}{\partial}}, \\ \left(1-\left(1-\prod_{ŧ=1}^{\Xi}\coprod_{a_ŧ{}'\in a_{pŧ}}\left(1-(1-a_ŧ{}'')^{\partial}\right)^{\Sigma_ŧ}\right)^{\frac{1}{\partial}}\right), \\ \left(1-\left(1-\prod_{ŧ=1}^{\Xi}\coprod_{f_ŧ{}'\in n_{pŧ}}\left(1-\left(1-f_ŧ{}''\right)^{\partial}\right)^{\Sigma_ŧ}\right)^{\frac{1}{\partial}}\right) \end{pmatrix}$$

So,
$\mathrm{PHFGWA}(\Gamma_1,\Gamma_2,..,\Gamma_\Xi) \ge \mathrm{PHFGWA}(\Gamma_1',\Gamma_2',..,\Gamma_\Xi')$.

Theorem 4: Let $\Gamma_j = \left\{\left(m_{pŧj}, a_{pŧj}, n_{pŧj}\right): \mathscr{v} \in X\right\}(ŧ = 1,2,..,\Xi)$ be the family of PHFNs lies between man and min operators with $\partial > 0$. Then

$$\min(\Gamma_1,\Gamma_2,..,\Gamma_\Xi) \le \mathrm{PHFGWA}(\Gamma_1,\Gamma_2,..,\Gamma_\Xi) \le \max(\Gamma_1,\Gamma_2,..,\Gamma_\Xi)$$

Proof: We know that $\alpha = \min(\Gamma_1,\Gamma_2,..,\Gamma_\Xi)$ and $\beta = \max(\Gamma_1,\Gamma_2,..,\Gamma_\Xi)$, then

$$\alpha \le \mathrm{PHFGWA}(\Gamma_1,\Gamma_2,..,\Gamma_\Xi) \le \beta$$

Then, we get

$$\left(\sum_{ŧ=1}^{\Xi}\Sigma_ŧ\alpha^{\partial}\right)^{\frac{1}{\partial}} \le \left(\sum_{ŧ=1}^{\Xi}\Sigma_ŧ\Xi_ŧ^{\partial}\right)^{\frac{1}{\partial}} \le \left(\sum_{ŧ=1}^{\Xi}\Sigma_ŧ\beta^{\partial}\right)^{\frac{1}{\partial}}$$

Implies that

$$\alpha \le \left(\sum_{ŧ=1}^{\Xi}\Sigma_ŧ\Xi_ŧ^{\partial}\right)^{\frac{1}{\partial}} \le \beta$$

That is

$$\min(\Gamma_1,\Gamma_2,..,\Gamma_\Xi) \le \mathrm{PHFGWA}(\Gamma_1,\Gamma_2,..,\Gamma_\Xi) \le \max(\Gamma_1,\Gamma_2,..,\Gamma_\Xi)$$

Hence the result is completed.

Some special cases concerning the parameter, such that

1. If $\partial\rightarrow 0$, the proposed AOs degenerate into PHF weighted geometric operator listed below:

   $$\mathrm{PHFWG}(\Gamma_1,\Gamma_2,..,\Gamma_\Xi) = \prod_{ŧ=1}^{\Xi}(\Gamma_ŧ)^{\Sigma_ŧ}$$

$$=\left(\begin{array}{c}\left(\prod_{ƚ=1}^{\Xi}\coprod_{t_ƚ'\in m_{pƚ}}(t_ƚ')^{\Sigma_ƚ}\right),\\ \left(1-\prod_{ƚ=1}^{\Xi}\coprod_{a_ƚ'\in a_{pƚ}}(1-a_ƚ')^{\Sigma_ƚ}\right),\\ \left(1-\prod_{ƚ=1}^{\Xi}\coprod_{f_ƚ'\in n_{pƚ}}(1-f_ƚ')^{\Sigma_ƚ}\right)\end{array}\right)$$

2. If $\partial = 1$, the proposed AOs degenerate into PHF weighted averaging operator:

$$\mathrm{PHFGWA}(\Gamma_1,\Gamma_2,..,\Gamma_\Xi)=\left(\sum_{ƚ=1}^{\Xi}\Sigma_ƚ\Gamma_ƚ\right)$$

$$=\left(\begin{array}{c}\left(1-\prod_{ƚ=1}^{\Xi}\coprod_{t_ƚ'\in m_{pƚ}}(1-t_ƚ')^{\Sigma_ƚ}\right),\\ \left(\prod_{ƚ=1}^{\Xi}\coprod_{a_ƚ'\in a_{pƚ}}(a_ƚ')^{\Sigma_ƚ}\right),\left(\prod_{ƚ=1}^{\Xi}\coprod_{f_ƚ'\in n_{pƚ}}(f_ƚ')^{\Sigma_ƚ}\right)\end{array}\right)$$

3. If $\partial = 2$, the proposed AOs degenerate into PHF weighted quadratic averaging operator:

$$\mathrm{PHFGWQA}(\Gamma_1,\Gamma_2,..,\Gamma_\Xi)=\left(\sum_{ƚ=1}^{\Xi}\Sigma_ƚ\Gamma_ƚ^{2}\right)^{\frac{1}{2}}$$

$$=\left(\begin{array}{c}\left(1-\prod_{ƚ=1}^{\Xi}\coprod_{t_ƚ'\in m_{pƚ}}(1-(t_ƚ')^2)^{\Sigma_ƚ}\right)^{\frac{1}{2}},\\ \left(1-\left(1-\prod_{ƚ=1}^{\Xi}\coprod_{a_ƚ'\in a_{pƚ}}(1-(1-a_ƚ')^2)^{\Sigma_ƚ}\right)^{\frac{1}{2}}\right),\\ \left(1-\left(1-\prod_{ƚ=1}^{\Xi}\coprod_{f_ƚ'\in n_{pƚ}}(1-(1-f_ƚ')^2)^{\Sigma_ƚ}\right)^{\frac{1}{2}}\right)\end{array}\right)$$

Definition 11: The PHFGOWA operator is given by: $\mathrm{PHFGOWA}:\Omega^{\Xi}\to\Omega$ by

$$\mathrm{PHFGOWA}(\Gamma_1,\Gamma_2,..,\Gamma_\Xi)=\left(\sum_{ƚ=1}^{\Xi}\Sigma_ƚ\Gamma_{o(ƚ)}^{\partial}\right)^{\frac{1}{\partial}}$$

Where $\Gamma_{o(ƚ)}\leq\Gamma_{o(ƚ-1)}$.

Theorem 5: Let $\Gamma_j=\{(m_{pƚj},a_{pƚj},n_{pƚj}):\mathcal{v}\in X\}(ƚ=1,2,..,\Xi)$ be the family of PHFNs with $\partial>0$. Then by using the theory of PHFGOWA, we again received PHFN, such that

$\mathrm{PHFGOWA}(\Gamma_1,\Gamma_2,..,\Gamma_\Xi)$

$$=\left(\begin{array}{c}\left(1-\prod_{ƚ=1}^{\Xi}\coprod_{t_ƚ'\in m_{pƚ}}\left(1-\left(t_{o(ƚ)}'\right)^{\partial}\right)^{\Sigma_ƚ}\right)^{\frac{1}{\partial}},\\ \left(1-\left(1-\prod_{ƚ=1}^{\Xi}\coprod_{a_ƚ'\in a_{pƚ}}\left(1-\left(1-a_{o(ƚ)}'\right)^{\partial}\right)^{\Sigma_ƚ}\right)^{\frac{1}{\partial}}\right),\\ \left(1-\left(1-\prod_{ƚ=1}^{\Xi}\coprod_{f_ƚ'\in n_{pƚ}}\left(1-\left(1-f_{o(ƚ)}'\right)^{\partial}\right)^{\Sigma_ƚ}\right)^{\frac{1}{\partial}}\right)\end{array}\right)$$

Proof: Omitted.

Next, some core properties of the initiated aggregation operators are investigated such as idempotency, monotonicity, and boundedness.

Theorem 6: Let $\Gamma_j=\{(m_{pƚj},a_{pƚj},n_{pƚj}):\mathcal{v}\in X\}(ƚ=1,2,..,\Xi)$ be the family of PHFNs with $\partial>0$. If $\Gamma=\Gamma_j$, then

$\mathrm{PHFGOWA}(\Gamma_1,\Gamma_2,..,\Gamma_\Xi)=\Gamma$

Proof: Omitted.

Theorem 7: Let $\Gamma_j=\{(m_{pƚj},a_{pƚj},n_{pƚj}):\mathcal{v}\in X\}(ƚ=1,2,..,\Xi)$ be the family of PHFNs with $\partial>0$. If $\Gamma_j\geq\Gamma_{j'}$. Then

$\mathrm{PHFGOWA}(\Gamma_1,\Gamma_2,..,\Gamma_\Xi)\geq\mathrm{PHFGOWA}(\Gamma_1',\Gamma_2',..,\Gamma_\Xi')$

Proof: Omitted.

Theorem 8: Let $\Gamma_j=\{(m_{pƚj},a_{pƚj},n_{pƚj}):\mathcal{v}\in X\}(ƚ=1,2,..,\Xi)$ be the family of PHFNs lies between man and min operators with $\partial>0$. Then

$$\min(\Gamma_1,\Gamma_2,..,\Gamma_\Xi)\leq\mathrm{PHFGOWA}(\Gamma_1,\Gamma_2,..,\Gamma_\Xi)\leq\max(\Gamma_1,\Gamma_2,..,\Gamma_\Xi)$$

Proof: We know that $\alpha=\min(\Gamma_1,\Gamma_2,..,\Gamma_\Xi)$ and $\beta=\max(\Gamma_1,\Gamma_2,..,\Gamma_\Xi)$, then

$\alpha\leq\mathrm{PHFGOWA}(\Gamma_1,\Gamma_2,..,\Gamma_\Xi)\leq\beta$

Then, we get

$$\left(\sum_{ƚ=1}^{\Xi}\Sigma_ƚ\alpha^{\partial}\right)^{\frac{1}{\partial}}\leq\left(\sum_{ƚ=1}^{\Xi}\Sigma_ƚ\Gamma_{o(ƚ)}^{\partial}\right)^{\frac{1}{\partial}}\leq\left(\sum_{ƚ=1}^{\Xi}\Sigma_ƚ\beta^{\partial}\right)^{\frac{1}{\partial}}$$

Implies that

$$\alpha\leq\left(\sum_{ƚ=1}^{\Xi}\Sigma_ƚ\Gamma_{o(ƚ)}^{\partial}\right)^{\frac{1}{\partial}}\leq\beta$$

That is

$$\min(\Gamma_1,\Gamma_2,..,\Gamma_\Xi)\leq\mathrm{PHFGOWA}(\Gamma_1,\Gamma_2,..,\Gamma_\Xi)\leq\max(\Gamma_1,\Gamma_2,..,\Gamma_\Xi).$$

Some special cases concerning the parameter, such that

1. If $\partial\to0$, the proposed AOs degenerate into PHF ordered weighted geometric operator:

$$\mathrm{PHFOWG}(\Gamma_1,\Gamma_2,..,\Gamma_\Xi)=\prod\nolimits_{ƚ=1}^{\Xi}\left(\Gamma_{o(ƚ)}\right)^{\Sigma_ƚ}$$

$$= \begin{pmatrix} \left(\prod_{\mathfrak{t}=1}^{\Xi} \coprod_{t_{\mathfrak{t}}{}' \in m_{p\mathfrak{t}}} \left(t_{o(\mathfrak{t})}{}'\right)^{\Sigma_{\mathfrak{t}}}\right), \\ \left(1-\prod_{\mathfrak{t}=1}^{\Xi} \coprod_{a_{\mathfrak{t}}{}' \in a_{p\mathfrak{t}}} \left(1-a_{o(\mathfrak{t})}{}'\right)^{\Sigma_{\mathfrak{t}}}\right), \\ \left(1-\prod_{\mathfrak{t}=1}^{\Xi} \coprod_{f_{\mathfrak{t}}{}' \in n_{p\mathfrak{t}}} \left(1-f_{o(\mathfrak{t})}{}'\right)^{\Sigma_{\mathfrak{t}}}\right) \end{pmatrix}$$

2. If $\partial = 1$, the proposed AOs degenerate into PHF ordered weighted averaging operator:

$$\mathrm{PHFGOWA}(\Gamma_1, \Gamma_2, \ldots, \Gamma_\Xi) = \left(\sum_{\mathfrak{t}=1}^{\Xi} \Sigma_{\mathfrak{t}} \Gamma_{o(\mathfrak{t})}\right)$$

$$= \begin{pmatrix} \left(1-\prod_{\mathfrak{t}=1}^{\Xi} \coprod_{t_{\mathfrak{t}}{}' \in \mathrm{T}_{\mathfrak{t}}} \left(1-t_{o(\mathfrak{t})}{}'\right)^{\Sigma_{\mathfrak{t}}}\right), \\ \left(\prod_{\mathfrak{t}=1}^{\Xi} \coprod_{a_{\mathfrak{t}}{}' \in \ddot{A}_{\mathfrak{t}}} \left(a_{o(\mathfrak{t})}{}'\right)^{\Sigma_{\mathfrak{t}}}\right), \\ \left(\prod_{\mathfrak{t}=1}^{\Xi} \coprod_{f_{\mathfrak{t}}{}' \in \mathrm{F}_{\mathfrak{t}}} \left(f_{o(\mathfrak{t})}{}'\right)^{\Sigma_{\mathfrak{t}}}\right) \end{pmatrix}$$

3. If $\partial = 2$, the proposed AOs degenerate into PHF ordered weighted quadratic averaging operator:

$$\mathrm{PHFGOWQA}(\Gamma_1, \Gamma_2, \ldots, \Gamma_\Xi) = \left(\sum_{\mathfrak{t}=1}^{\Xi} \Sigma_{\mathfrak{t}} \Gamma_{o(\mathfrak{t})}^2\right)^{\frac{1}{2}}$$

$$= \begin{pmatrix} \left(1-\prod_{\mathfrak{t}=1}^{\Xi} \coprod_{t_{\mathfrak{t}}{}' \in m_{p\mathfrak{t}}} \left(1-\left(t_{o(\mathfrak{t})}{}'\right)^2\right)^{\Sigma_{\mathfrak{t}}}\right)^{\frac{1}{2}}, \\ \left(1-\left(1-\prod_{\mathfrak{t}=1}^{\Xi} \coprod_{a_{\mathfrak{t}}{}' \in a_{p\mathfrak{t}}} \left(1-\left(1-a_{o(\mathfrak{t})}{}'\right)^2\right)^{\Sigma_{\mathfrak{t}}}\right)^{\frac{1}{2}}\right), \\ \left(1-\left(1-\prod_{\mathfrak{t}=1}^{\Xi} \coprod_{f_{\mathfrak{t}}{}' \in n_{p\mathfrak{t}}} \left(1-\left(1-f_{o(\mathfrak{t})}{}'\right)^2\right)^{\Sigma_{\mathfrak{t}}}\right)^{\frac{1}{2}}\right) \end{pmatrix}$$

Definition 12: The PHFGHWA operator is given by: PHFGHWA: $\Omega^{\Xi} \to \Omega$ by

$$\mathrm{PHFGHWA}(\Gamma_1, \Gamma_2, \ldots, \Gamma_\Xi) = \left(\sum_{\mathfrak{t}=1}^{\Xi} \Sigma_{\mathfrak{t}} \dot{\Gamma}_{o(\mathfrak{t})}^{\partial}\right)^{\frac{1}{\partial}}$$

Where $\Gamma_{o(\mathfrak{t})} \leq \Gamma_{o(\mathfrak{t}-1)}$.

Theorem 9: Let $\Gamma_j = \left\{\left(m_{p\mathfrak{t}j}, a_{p\mathfrak{t}j}, n_{p\mathfrak{t}j}\right): \mathscr{v} \in X\right\}(\mathfrak{t} = 1,2,\ldots,\Xi)$ be the family of PHFNs with $\partial > 0$. Then by using the theory of PHFGHWA, we again received PHFN, such that

$$\mathrm{PHFGHWA}(\Gamma_1, \Gamma_2, \ldots, \Gamma_\Xi)$$

$$= \begin{pmatrix} \left(1-\prod_{\mathfrak{t}=1}^{\Xi} \coprod_{t_{\mathfrak{t}}{}' \in m_{p\mathfrak{t}}} \left(1-\left(\dot{t}_{o(\mathfrak{t})}{}'\right)^{\partial}\right)^{\Sigma_{\mathfrak{t}}}\right)^{\frac{1}{\partial}}, \\ \left(1-\left(1-\prod_{\mathfrak{t}=1}^{\Xi} \coprod_{a_{\mathfrak{t}}{}' \in a_{p\mathfrak{t}}} \left(1-\left(1-\dot{a}_{o(\mathfrak{t})}{}'\right)^{\partial}\right)^{\Sigma_{\mathfrak{t}}}\right)^{\frac{1}{\partial}}\right), \\ \left(1-\left(1-\prod_{\mathfrak{t}=1}^{\Xi} \coprod_{f_{\mathfrak{t}}{}' \in n_{p\mathfrak{t}}} \left(1-\left(1-\dot{f}_{o(\mathfrak{t})}{}'\right)^{\partial}\right)^{\Sigma_{\mathfrak{t}}}\right)^{\frac{1}{\partial}}\right) \end{pmatrix}$$

Proof: Omitted.

Next, some core properties of the initiated aggregation operators are investigated such as idempotency, monotonicity, and boundedness.

Theorem 10: Let $\Gamma_j = \left\{\left(m_{p\mathfrak{t}j}, a_{p\mathfrak{t}j}, n_{p\mathfrak{t}j}\right): \mathscr{v} \in X\right\}(\mathfrak{t} = 1,2,\ldots,\Xi)$ be the family of PHFNs with $\partial > 0$. If $\Gamma = \Gamma_j$, then

$\mathrm{PHFGHWA}(\Gamma_1, \Gamma_2, \ldots, \Gamma_\Xi) = \Gamma$

Proof: Omitted.

Theorem 11: Let $\Gamma_j = \left\{\left(m_{p\mathfrak{t}j}, a_{p\mathfrak{t}j}, n_{p\mathfrak{t}j}\right): \mathscr{v} \in X\right\}(\mathfrak{t} = 1,2,\ldots,\Xi)$ be the family of PHFNs with $\partial > 0$. If $\Gamma_j \geq \Gamma_{j'}$. Then

$\mathrm{PHFGHWA}(\Gamma_1, \Gamma_2, \ldots, \Gamma_\Xi) \geq \mathrm{PHFGHWA}(\Gamma_1', \Gamma_2', \ldots, \Gamma_\Xi')$

Proof: Omitted.

Theorem 12: Let $\Gamma_j = \left\{\left(m_{p\mathfrak{t}j}, a_{p\mathfrak{t}j}, n_{p\mathfrak{t}j}\right): \mathscr{v} \in X\right\}(\mathfrak{t} = 1,2,\ldots,\Xi)$ be the family of PHFNs lies between man and min operators with $\partial > 0$. Then

$$\min(\Gamma_1, \Gamma_2, \ldots, \Gamma_\Xi) \leq \mathrm{PHFGHWA}(\Gamma_1, \Gamma_2, \ldots, \Gamma_\Xi) \leq \max(\Gamma_1, \Gamma_2, \ldots, \Gamma_\Xi)$$

Proof: We know that $\alpha = \min(\Gamma_1, \Gamma_2, \ldots, \Gamma_\Xi)$ and $\beta = \max(\Gamma_1, \Gamma_2, \ldots, \Gamma_\Xi)$, then

$\alpha \leq \mathrm{PHFGHWA}(\Gamma_1, \Gamma_2, \ldots, \Gamma_\Xi) \leq \beta$

Then, we get

$$\left(\sum_{\mathfrak{t}=1}^{\Xi} \Sigma_{\mathfrak{t}} \alpha^{\partial}\right)^{\frac{1}{\partial}} \leq \left(\sum_{\mathfrak{t}=1}^{\Xi} \Sigma_{\mathfrak{t}} \dot{\Gamma}_{o(\mathfrak{t})}^{\partial}\right)^{\frac{1}{\partial}} \leq \left(\sum_{\mathfrak{t}=1}^{\Xi} \Sigma_{\mathfrak{t}} \beta^{\partial}\right)^{\frac{1}{\partial}}$$

Implies that

$$\alpha \leq \left(\sum_{\mathfrak{t}=1}^{\Xi} \Sigma_{\mathfrak{t}} \dot{\Gamma}_{o(\mathfrak{t})}^{\partial}\right)^{\frac{1}{\partial}} \leq \beta$$

That is

$$\min(\Gamma_1, \Gamma_2, \ldots, \Gamma_\Xi) \leq \mathrm{PHFGHWA}(\Gamma_1, \Gamma_2, \ldots, \Gamma_\Xi) \leq \max(\Gamma_1, \Gamma_2, \ldots, \Gamma_\Xi).$$

Some special cases concerning the parameter, such that

1. If $\partial \to 0$, the proposed AOs degenerate into PHF hybrid weighted geometric (PHFHWG) operator:

ISSN:1813-1786 (Print) 2313-7770 (Online)

$$\text{PHFHWG}(\Gamma_1,\Gamma_2,..,\Gamma_\Xi)=\prod_{ŧ=1}^{\Xi}\left(\dot{\Gamma}_{o(ŧ)}\right)^{\Sigma_ŧ}$$
$$=\begin{pmatrix}\left(\prod_{ŧ=1}^{\Xi}\coprod_{t_ŧ{}'\in m_{pŧ}}\left(\dot{t}_{o(ŧ)}{}'\right)^{\Sigma_ŧ}\right),\\ \left(1-\prod_{ŧ=1}^{\Xi}\coprod_{a_ŧ{}'\in a_{pŧ}}\left(1-\dot{a}_{o(ŧ)}{}'\right)^{\Sigma_ŧ}\right),\\ \left(1-\prod_{ŧ=1}^{\Xi}\coprod_{f_ŧ{}'\in n_{pŧ}}\left(1-\dot{f}_{o(ŧ)}{}'\right)^{\Sigma_ŧ}\right)\end{pmatrix}$$

2. If $\partial=1$, the proposed AOs degenerate into PHF hybrid weighted averaging (PHFHWA) operator:

$$\text{PHFGHWA}(\Gamma_1,\Gamma_2,..,\Gamma_\Xi)=\left(\sum_{ŧ=1}^{\Xi}\Sigma_ŧ\dot{\Gamma}_{o(ŧ)}\right)$$
$$=\begin{pmatrix}\left(1-\prod_{ŧ=1}^{\Xi}\coprod_{t_ŧ{}'\in m_{pŧ}}\left(1-\dot{t}_{o(ŧ)}{}'\right)^{\Sigma_ŧ}\right),\\ \left(\prod_{ŧ=1}^{\Xi}\coprod_{a_ŧ{}'\in a_{pŧ}}\left(\dot{a}_{o(ŧ)}{}'\right)^{\Sigma_ŧ}\right),\\ \left(\prod_{ŧ=1}^{\Xi}\coprod_{f_ŧ{}'\in n_{pŧ}}\left(\dot{f}_{o(ŧ)}{}'\right)^{\Sigma_ŧ}\right)\end{pmatrix}$$

3. If $\partial=2$, the proposed AOs degenerate into PHF hybrid weighted quadratic averaging (PHFHWQA) operator:

$$\text{PHFGHWQA}(\Gamma_1,\Gamma_2,..,\Gamma_\Xi)=\left(\sum_{ŧ=1}^{\Xi}\Sigma_ŧ\dot{\Gamma}_{o(ŧ)}^2\right)^{\frac{1}{2}}$$
$$=\begin{pmatrix}\left(1-\prod_{ŧ=1}^{\Xi}\coprod_{t_ŧ{}'\in Ŧ_ŧ}\left(1-\left(\dot{t}_{o(ŧ)}{}'\right)^2\right)^{\Sigma_ŧ}\right)^{\frac{1}{2}},\\ \left(1-\left(1-\prod_{ŧ=1}^{\Xi}\coprod_{a_ŧ{}'\in Ä_ŧ}\left(1-\left(1-\dot{a}_{o(ŧ)}{}'\right)^2\right)^{\Sigma_ŧ}\right)^{\frac{1}{2}}\right),\\ \left(1-\left(1-\prod_{ŧ=1}^{\Xi}\coprod_{f_ŧ{}'\in Ƒ_ŧ}\left(1-\left(1-\dot{f}_{o(ŧ)}{}'\right)^2\right)^{\Sigma_ŧ}\right)^{\frac{1}{2}}\right)\end{pmatrix}$$

IV. MADM METHOD FOR PRESENTED INFORMATION

The Decision-making technique contained four basic and valuable parts, called MADM technique, multicriteria decision-making, multi-attribute group decision-making, and multicriteria group decision-making techniques which are used for obtaining beneficial and valuable information from the family of preferences. Certain people have utilized the environment of different fields under the consideration of FSs, IFSs, PFSs, HFSs, and PHFSs. Further, we know that without a decision-making technique it is very awkward to find our required information, therefore, in this manuscript, we also evaluated a MADM procedure under the availability of diagnosed operators to show the supremacy and worth of the pioneered information. Finally, we aim to evaluate the comparative analysis of the pioneered information with some prevailing operators to enhance the capability and accuracy of the proposed information.

The aim of this study, we consider the established methods for the PHFNs to the MADM problems. The MADM process is followed as $X=\{\mathcal{v}_1,\mathcal{v}_2,\dots,\mathcal{v}_m\}$ represents the set of alternatives and the set of attributes takes the form of PHFNs represented by $C=\{c_1,c_2,\dots,c_\Xi\}$ concerning weight, represented by: by $\Sigma=(\Sigma_1,\Sigma_2,\dots,\Sigma_\Xi)^Ŧ,\sum_{ŧ=1}^{\Xi}\Sigma_ŧ=1$. Consider the family of PHFNs $\Gamma_j=\{(m_{pŧj},a_{pŧj},n_{pŧj}):\mathcal{v}\in X\}(ŧ,j=1,2,..,\Xi)$ of the criteria $C_ŧ$ on the alternative $Ä_ŧ$, which is representing the PHFNs. To evaluate the above dilemma, we used the following stages, such that

1. Compute decision information with the help of PHFNs.
2. Collect information in the shape of PHFNs, then we try to aggregate the information by using the PHFGHA operator to aggregate all PHFNs into a single set.
3. Use the theory of score value $Ş(\Gamma)$, for evaluating the aggregate information into a single number.
4. Rank all the alternatives and examine the best one.
5. End.

Example 2: An organization is a legitimate substance framed by a gathering of people to participate in and work a business — business or modern — endeavor. An organization might be coordinated in different ways for charge and monetary risk purposes relying upon the corporate law of its purview. We will derive the numerical example with the help of proposed operators. The investment enterprises want to invest with a possible enterprise from all alternatives. The four candidates are marked by $Ä_ŧ$ and they are examining the three attributes:

1. C_1: Likelihood Index.
2. C_2: Extension Index.
3. C_3: Political Impact Index.

We will consider the weighted vectors $\Sigma=(0.4,0.2,0.4)^Ŧ$ and illustrate the following matrix for calculations. To evaluate the above dilemma, we used the following stages, such that

1. Compute decision information with the help of PHFNs, as stated in Table 1.

Table 4.1. Picture hesitant fuzzy generalized aggregation operator decision matrix.

	$\boldsymbol{C_1}$	$\boldsymbol{C_2}$	$\boldsymbol{C_3}$
$\ddot{A}_1$	$\begin{pmatrix}(0.3,0.2,0.1),\\(0.4,0.3,0.2),\\(0.5,0.2,0.1)\end{pmatrix}$	$\begin{pmatrix}(0.13,0.12,0.11),\\(0.14,0.13,0.12),\\(0.15,0.12,0.11)\end{pmatrix}$	$\begin{pmatrix}(0.3,0.2,0.1),\\(0.4,0.3,0.2),\\(0.5,0.2,0.1)\end{pmatrix}$
$\ddot{A}_2$	$\begin{pmatrix}(0.31,0.21,0.11),\\(0.41,0.31,0.21),\\(0.51,0.21,0.11)\end{pmatrix}$	$\begin{pmatrix}(0.23,0.22,0.21),\\(0.24,0.23,0.22),\\(0.25,0.22,0.21)\end{pmatrix}$	$\begin{pmatrix}(0.31,0.21,0.11),\\(0.41,0.31,0.21),\\(0.51,0.21,0.11)\end{pmatrix}$
$\ddot{A}_3$	$\begin{pmatrix}(0.32,0.22,0.12),\\(0.42,0.32,0.22),\\(0.52,0.22,0.12)\end{pmatrix}$	$\begin{pmatrix}(0.33,0.32,0.31),\\(0.34,0.33,0.32),\\(0.35,0.32,0.31)\end{pmatrix}$	$\begin{pmatrix}(0.13,0.12,0.11),\\(0.14,0.13,0.12),\\(0.15,0.12,0.11)\end{pmatrix}$
$\ddot{A}_4$	$\begin{pmatrix}(0.33,0.23,0.13),\\(0.43,0.33,0.23),\\(0.53,0.23,0.13)\end{pmatrix}$	$\begin{pmatrix}(0.43,0.42,0.1),\\(0.44,0.43,0.02),\\(0.45,0.2,0.1)\end{pmatrix}$	$\begin{pmatrix}(0.23,0.22,0.21),\\(0.24,0.23,0.22),\\(0.25,0.22,0.21)\end{pmatrix}$

2. Collect information in the shape of PHFNs, then we try to aggregate the information by using the PHFGHA operator to aggregate all PHFNs into a single set.
3. Use the theory of score value $Ş(\Gamma)$, for evaluating the aggregate information into a single number, $\acute{\omega} = (0.5,0.4,0.1)^{Ŧ}$ for $\partial = 1$, stated in Table 2.

Table 4.2. Ranking results of the PHFSs

Alternatives	***Şcore function***	***Rank***
$\ddot{A}_1$	$Ş(\ddot{A}_1) = 0.47$	4
$\ddot{A}_2$	$Ş(\ddot{A}_2) = 0.55$	1
$\ddot{A}_3$	$Ş(\ddot{A}_3) = 0.54$	2
$\ddot{A}_4$	$Ş(\ddot{A}_4) = 0.48$	3

4. Rank all the alternatives $\ddot{A}_{ŧ}(ŧ = 1,2,3,4)$ in descending order and find the best one such as: $\ddot{A}_2 \geq \ddot{A}_3 \geq \ddot{A}_4 \geq \ddot{A}_1$

So $\ddot{A}_2$ is the best one for investment.

V. Advantages and Comparative Analysis of the PHFSs

To evaluate the beneficial and valuable decision from the collection of preferences, the MADM technique is more effective for evaluating our main problem. Moreover, the main theory of PHF information is the modified and generalized version of the fuzzy, intuitionistic, picture, hesitant fuzzy information and some other extensions, and due to these reasons, they receive a lot of attention from different scholars. The major influence of this theory is to comparative analysis of the pioneered information with some prevailing operators to enhance the capability and accuracy of the proposed information. To compare the diagnosed operator with some exiting operators is to enhance the quality of the evaluated information, for this, we use the theory of Ullah et a. [36] and the theory of Wang and Li [37]. Finally, the comparative analysis of the proposed work with existing works is available in Table 3.

From the information discussed in Table 3, we clear that the operators diagnosed by Ullah et a. [36], the theory of Wang and Li [37], and the proposed information are given the same ranking result $\ddot{A}_2$. Using the information in Table 3, we have drawn their geometrical shape in the shape of Figure 1.

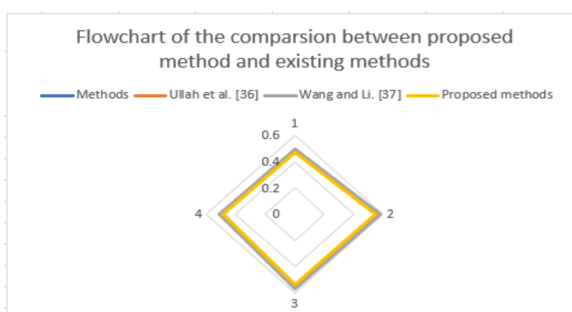

Fig. 1. Comparison between proposed method with existing methods.

Table 4.3. Score value and ranking values using Example 1.

Methods	Using the formula of the Score values	Ranking Results of the different methods	Best alternatives
Ullah et al. [36]	$\c{S}(\ddot{A}_1) = 0.48, \c{S}(\ddot{A}_2) = 0.56,$ $\c{S}(3) = 0.55, \c{S}(\ddot{A}_4) = 0.49$	$\ddot{A}_2 \geq \ddot{A}_3 \geq \ddot{A}_4 \geq \ddot{A}_1$	$\ddot{A}_2$
Wang and Li. [37]	$\c{S}(\ddot{A}_1) = 0.50, \c{S}(\ddot{A}_2) = 0.58,$ $\c{S}(3) = 0.57, \c{S}(\ddot{A}_4) = 0.51$	$\ddot{A}_2 \geq \ddot{A}_3 \geq \ddot{A}_4 \geq \ddot{A}_1$	$\ddot{A}_2$
Proposed methods	$\c{S}(\ddot{A}_1) = 0.47, \c{S}(\ddot{A}_2) = 0.55,$ $\c{S}(3) = 0.54, \c{S}(\ddot{A}_4) = 0.48$	$\ddot{A}_2 \geq \ddot{A}_3 \geq \ddot{A}_4 \geq \ddot{A}_1$	$\ddot{A}_2$

Therefore, the pioneered operators are massively useful and valuable as compared to prevailing operators.

VI. CONCLUSION

To evaluate the beneficial and valuable decision from the collection of preferences, the MADM technique is more effective for evaluating our main problem. Moreover, the main theory of PHF information is the modified and generalized version of the fuzzy, intuitionistic, picture, hesitant fuzzy information and some other extensions, and due to these reasons, they receive a lot of attention from different scholars. The major influence of this theory is discussed below:

1. We pioneered the theory of generalized hybrid aggregation operators under the consideration of PHF information, called PHFGWA, PHFGOWA, and PHFGHWA operators.
2. We evaluated their valuable and flexible properties with some dominant results.
3. We discussed a MADM procedure under the availability of diagnosed operators to show the supremacy and worth of the pioneered information.
4. We evaluated the comparative analysis of the pioneered information with some prevailing operators to enhance the capability and accuracy of the proposed information.

VII. ADVANTAGES AND DISADVANTAGES

Operators defined based on PHFS are more powerful because the theory of PHFSs is the generalized shape of FSs, IFSs, PFSs, HFSs, and IHFSs. It means that by implementing different rules we can easily obtain the prevailing information defined based on FSs, IFSs, PFSs, HFSs, and IHFSs. But in various situations, the theory defined based on PHFS has failed, due to some complications, because in the presence of spherical hesitant fuzzy sets and T-spherical hesitant fuzzy sets, the operators based on PHFS have been neglected.

VIII. FUTURE DIRECTIONS

In the future, we will utilize the proposed concept in the environment of spherical fuzzy sets [42-43], picture fuzzy sets [44], and also study the complex PHFS. In the following articles, we will convert it for established work [45-46].

IX. ACKNOWLEDGMENT

The authors declare that there is no conflict of interest.

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