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# Balanced Truncation of Stable Discrete Time Second-Orderly Structured Systems

S. Haider[1], S. Ali[2], H. Muazzam[3], R. M. Mokhtar[4], K. Saleem[5]

[1,2,3,4]*Universiti Sains Malaysia, 11800 USM Penang, Malaysia*
[5]*Graduate School of Sciences and Engineering Koc University, Istanbul, Turkey*

[2]sadaqat.ali@student.usm.my

***Abstract-*** Mathematical modeling is the only depiction of the physical systems in literary world. But due to complexity, these models are to be expressed in equivalently curtailed models. Framework for sundry non-existent techniques for order curtailment of stable discrete time second-orderly structured systems (SOSs) are suggested in this manuscript. Discrete SOSS is transmuted into generalized configuration and discrete time algebraic Lyapunov equations (DALEs) are composed. Gramians for balancing are procured from DALEs. Fragmentation of gramians into position and velocity snippets is of utmost importance. Once fragmentation takes place, structure retention in ROM is procured and balanced curtailment is solicited. Sundry versions of balanced transformations are introduced. Position and velocity gramians are balanced in disparate combinations. Balanced truncation based on magnitude of Hankel singular values is established to procure the reduced order model that veraciously depict the incipient system. As suggested contrivance hangs on to second-orderly structure and carries dynamics of stabilized system therefore, this contrivance truely surmises original system behavior. Numerical results are incorporated to ratify the suggested mechanism.

***Keywords-*** Second-orderly structured systems, unstable systems, model order reduction, infinite gramians, Hankel singular values.

## I. INTRODUCTION

The mathematical model of linear time invarient discrete time system is derived for analytical and delineation purposes of system. The reduced order models (ROM) of these mathematical models are highly demanded becauseof complexity of these models. ROM is a surrogate model of the original model and second-orderly structured systems hold pair of 1st and 2nd state-derivatives in system-dynamics. These systems are found in multifarious fields related to engineering and technology like biological systems, electro-mechanical technology, image processing community interaction, smart grid systems, huge-buildings, [1–6] etc. The order of differential equations of such system models is large (in billions) and it turns out to be rather tough to comprehend and analyse such systems as processing and corresponding storage issues are grave in nature. In such scenarios, model order reduction (MOR) contrivances are formulated that minutely surmise complex system characteristics with lower order surrogates that can be carry through. In ROM, some characteristics of complex system need to be preserved. Such characteristics comprise passivity, stability and regularity etc. Curtailment error should be at certain desirable level and MOR contrivance must be efficacious and concentering. The linear time invariant SOSS in discrete shape is emblematized in (1).

$$Mx[k+2] + Dx[k+1] + Kx[k] = B_2u[k]$$
$$C_2x[k+1] + C_1x[k] = y[k] \qquad (1)$$

In generalized state space form, (1) can be written as [5]:

$$Eq[k+1] = Aq[k] + Bu[k]$$
$$y[k] = Cq[k] \qquad (2)$$

where $q[k] = [x[k]^T \ \ x[k+1]^T]^T$,

$$E = \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}, \ A = \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix}, B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix},$$
$$C = [C_1 \quad C_2]$$

$M \in R^{nxn}, D \in R^{nxn}, \ K \in R^{nxn}, B_2 \in R^{nxm}$, $C_1$ and $C_2 \in R^{pxn}, x[k] \in R^n, u[k] \in R^m$ and $y[k] \in R^p$.

## II. RELATED WORK

First method for model order reduction (MOR) was introduced by [7] in 1966 for generalized systems which was further modified by [8], [9]. Benner Structure-retaining extensions of classical model curtailment methods such as dominant pole contrivance, modal curtailment [10], moment-matching [11], balanced curtailment [12], or for example of the H2-optimal iterative rational Krylov contrivance [13]. In [14], author introduced an balancing concept that is strongly related to the origins of balanced curtailment than the other extensions. In the context of first order systems, confabulations in [15], are aimed at such local surmises. Nevertheless, earlier curtailment approaches did not retain the structure in ROM [16], [17]. But later developed MOR contrivances like modified Arnoldi, second-orderly structured balanced truncation method (SOBTM) and moment-matching (based on Krylov-subspaces) [2], [11], [14], [17]–[21] etc. retained second-orderly structure in ROM. In Krylov, Arnoldi, or moment-matching contrivances, although, curtailment of the model was achieved but ROM got unstable in the due course. Accordingly, a-priori curtailment error bounds do not exist. But stability of ROM is retained and so exist the a-priori curtailment error-bound for SOBTM [16], [17], [22]–[24]. [12] portrayed multifarious SOBTM contrivances for stable SOSSs. Contemplating, the said research gap, in this publication, structure retaining MOR contrivances for stable SOSSs are initiated. The SOSS is supplicated in CALEs to procure gramians for balanced curtailment. A methodical and coherent scheme is invoked to procure Cholesky factorization of LFGs and solution of CALEs. Methodical gramians are flaked in position and velocity snippets to ensure structure retention in ROM. Moreover, the procured gramians are balanced with different coalescences to find sundry SOBTM contrivances. Curtailment relents the desired ROM that exhibit SOSS exposition as well as optimized accomplishment. Suggested contrivances are collocated with prevailing gramians contrivance for sundry unstable SOSSs to endorse the veracious development and dominance. Sequels for few examples are depicted and accomplished juxtaposition of suggested contrivances is discussed. Manuscript is arranged as follows. Next section discusses preliminaries on stable SOSSs followed by suggested contrivances in section 4. Results are discussed in section 5 and conclusion is proposed.

## III. SECOND ORDERLY STRUCTURED SYSTEMS

The transfer matrix of SOS (1) is given by:

$$H(z) = (zC_2 + C_1)(z^2M + zD + K)^{-1}B_2 \tag{3}$$

and is represented in short as

$H = [M, D, K, B_2, C_1, C_2]$.

System (1) is stable if all zeros of $P(\lambda) = \lambda^2 M + \lambda D + K$ lie within the unit circle. Moreover, system (1) is controllable if

$$rank[\lambda^2 M + \lambda D + K, B2] = n \quad for\ all\ \ \lambda \epsilon C$$

and is observable if

$$rank[\lambda^2 M^T + \lambda D^T + K^T, \lambda C_2^T + C_1^T] = n$$

Correspondingly, system (2) is stable if all the eigenvalues (EVs) of the pencil $\lambda E - A$ lie within the unit circle, system (2) is controllable and observable if and only if the first order system (2) is controllable and observable respectively i.e. $rank[\lambda E - A, B] = 2n$ and $rank[\lambda E^T - A^T, C^T] = 2n\ for\ all\ \lambda \epsilon C.$

The goal of MOR technique is to produce a ROM given as:

$$M_r x_r[k+2] + D_r x_r[k+1] + K_r x_r[k] = B_{2r} u[k]$$
$$C_{2r} x_r[k+1] + C_{1r} x_r[k] = y_r[k] \tag{4}$$

where

$M_r, D_r,$

$K_r \epsilon R^{rxr}, B_{2r} \epsilon R^{rxm}, C_{1r}\ and\ C_{2r} \epsilon R^{pxr}$, such that $r << n$. Corresponding ROM (4) in first order generalized structure becomes:

$$E_r q_r[k+1] = A_r q_r[k] + B_r u[k]$$
$$y_r[k] = C_r q_r[k]$$

Using a system equivalence transformation $(P_l, P_r)$ with $P_l$ and $P_r$ being nonsingular, discrete system (1) can be transformed into an equivalent form:

$$\bar{M} = P_l M P_r, \quad \bar{D} = P_l D P_r, \quad \bar{K} = P_l K P_r$$
$$\bar{B}_2 = P_l B_2, \ \overline{C_1} = C_1 P_r, \ \overline{C_2} = C_2 P_r$$

and corresponding first order form (2) becomes:

$$\bar{E} = P_l E P_r, \ \bar{A} = P_l A P_r, \ \bar{B} = P_l B, \ \bar{C} = C P_r$$

with $P_l = \begin{bmatrix} P_r^{-1} & 0 \\ 0 & P_l \end{bmatrix}, P_r = \begin{bmatrix} P_r & 0 \\ 0 & P_r \end{bmatrix}.$

For stable system (2), controllability and observability Gramians defined as:

$$W_c = \sum_{k=0}^{\infty} F_d^k B B^T \left(F_d^k\right)^T \tag{5}$$
$$W_o = \sum_{k=0}^{\infty} (F_d^T)^k C^T C\, F_d \tag{6}$$

are the positive semi definite, unique and symmetric solutions of following DALEs:

$$A^T W_c A - E^T W_C E^T = -BB^T, \tag{7}$$
$$A W_o A^T + E W_o E^T = -C^T C \tag{8}$$

Where $F_d = e^{E^{-1}Ak} E^{-1}$ is the fundamental solution matrix of (2). The Gramians of (5) and (6) can be partitioned as

$$W_c = \begin{bmatrix} W_{pc} & W_{12c} \\ W_{12c}^T & W_{vc} \end{bmatrix}, W_o = \begin{bmatrix} W_{po} & W_{12o} \\ W_{12o}^T & W_{vo} \end{bmatrix}$$

where $W_{pc}$ and $W_{vc}$ are respectively the position and velocity controllability Gramians and $W_{po}$ and $W_{vo}$ are position and velocity observability Gramians of (2). The square root of EVs of the products $W_{pc}W_{po}$, $W_{vc}M^T W_{vo} M$, $W_{pc} M^T W_{vo} M$ and $W_{vc} W_{po}$ are the position, velocity, position-velocity and velocity-position Hankel singular values (HSVs) of SOS (2) respectively [5].

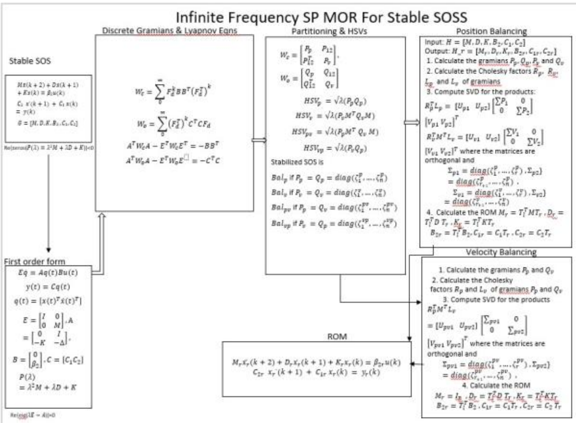


FIGURE 1: Methodology Flow Diagram

## IV. Suggested Methodology

Once transformed form is established, two algorithms for curtailment purposes are suggested

### *3.1. Balanced Transformation*

To compute position and velocity HSVs for balanced truncation, different balancing schemes can be applied to system (2).

*Definition 1:* The stable system (2) is called:

1. Position balanced if $W_{pc} = W_{po} = diag(\xi_1^p, \dots, \xi_n^p)$.
2. Velocity balanced if $W_{vc} = W_{vo} = diag(\xi_1^v, \dots, \xi_n^v)$.
3. Position-velocity balanced if $W_{pc} = W_{vo} = diag(\xi_1^{pv}, \dots, \xi_n^{pv})$.
4. Velocity-position balanced if $W_{vc} = W_{po} = diag(\xi_1^{vp}, \dots, \xi_n^{vp})$.

where $\xi$ represent position or velocity or both position-velocity (velocity-position) HSVs arranged in descending order.

In order to derive balanced transformation, consider the Cholesky factorization of Gramians:

$$W_{pc} = R_p R_p^T, \qquad W_{vc} = R_v R_v^T$$
$$W_{po} = L_p L_p^T, \qquad W_{vo} = L_v L_v^T$$

where $R_v, R_p, L_v, L_p \in R^{nxn}$ are Cholesky factors of Gramians that are used to compute HSVs as follows:

$$(\xi_i^p)^2 = \lambda_i(W_{pc} W_{po}) = \lambda_i(R_p R_p^T L_p L_p^T)) = \lambda_i(L_p^T R_p R_p^T L_p)$$
$$= \sigma_i^2(RpLp)$$

where $\sigma_i$'s are classical SVs. Similarly $\xi_i^v = \sigma_i(R_v^T M^T L_v)$, $\xi_{pv}^i = \sigma_i(R_p^T M^T L_v)$ $and$ $\xi_{vp}^i = \sigma_i(R_v^T L_p)$. Computing the singular value decomposition of these products give:

$$R_p^T L_p = U_p S_p V_p^T, \qquad R_v^T M^T L_v = U_v S_v V_v^T$$
$$R_p^T M^T L_v = U_{pv} S_{pv} V_{pv}^T, \quad R_v^T L_p = U_{vp} S_{vp} V_{vp}^T \qquad (9)$$

where $U_p, U_v, U_{pv}, U_{vp}, V_p, V_v, V_{pv}, V_{vp}$ are orthogonal matrices and

$$\Sigma_p = diag(\xi_1^p, \dots, \xi_n^p), \quad \Sigma_v = diag(\xi_1^v, \dots, \xi_n^v)$$
$$\Sigma_{pv} = diag(\xi_1^{pv}, \dots, \xi_n^{pv}), \quad \Sigma_{vp} = diag(\xi_1^{vp}, \dots, \xi_n^{vp})$$

are nonsingular matrices. Relation (9) can be used to determine the balanced transformation $(Pl, Pr)$ for schemes mentioned in definition 1.

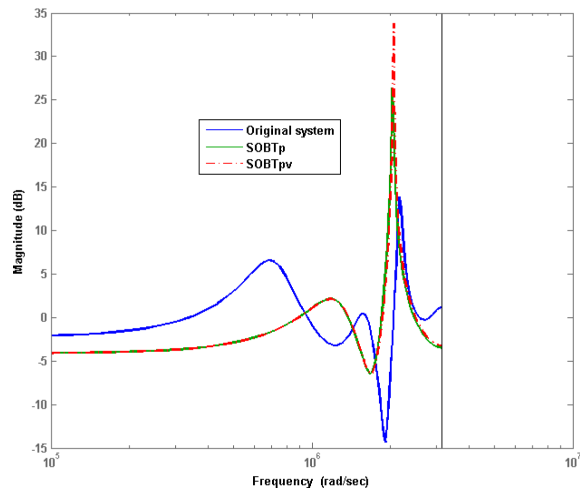


FIGURE 2: ROM response along with actual systemresponse, example 1

### *3.2. Second Order Balanced Truncation*

The balanced transformation of section 3.1 yield HSVs whose magnitudes depict the level of involvement of position and velocity states in system dynamics. States corresponding to small magnitudes are least involved and are truncated at small reduction error cost. Based on prior discussion in this section, in following, two algorithms for position and velocity balanced truncation are presented.

*Algorithm 1: Position Balancing based Discrete time SOBT (SOBTp)*

*Input:* Given a discrete time stable SOS $H(z)$

*Output:* The ROM $H_r(z)$

1. Calculate the Gramians $W_{pc}, W_{po}, W_{vc}$ $and$ $W_{vo}$ using (7) and (8).
2. Calculate the Cholesky factors $R_p, R_v, L_p$ $and$ $L_v$ of Gramians of step 1.
3. Compute SVD for the products:

$$R_p^T L_p = [U_{p1} \quad U_{p2}] \begin{bmatrix} \Sigma_{p1} & 0 \\ 0 & \Sigma_{p2} \end{bmatrix} [V_{p1} \quad V_{p2}]^T$$
$$R_v^T M^T L_v = [U_{v1} \quad U_{v2}] \begin{bmatrix} \Sigma_{v1} & 0 \\ 0 & \Sigma_{v2} \end{bmatrix} [V_{v1} \quad V_{v2}]^T$$

where the matrices $[U_{p1} \quad U_{p2}]$, $[V_{p1} \quad V_{p2}]$, $[U_{v1} \quad U_{v2}]$ and $[V_{v1} \quad V_{v2}]$ are orthogonal and $\Sigma_{p1} =$

$diag(\xi_1^p, \dots, \xi_r^p), \Sigma_{p2} = diag(\xi_{r+1}^p, \dots, \xi_n^p), \Sigma_{v1} = diag(\xi_1^v, \dots, \xi_r^v)$, $\Sigma_{v2} = diag(\xi_{r+1}^v, \dots, \xi_n^v)$.

4. Calculate the ROM:

$$M_r = P_l^T M P_r, D_r = D_l^T D P_r,$$
$$K_r = P_l^T K P_r, B_{2r} = P_l^T B_2, C_{1r} = C_1 P_r, \quad C_{2r} = C_2 P_r$$

Where $P_l = L_v V_{v1} \Sigma_{p1}^{-1/2}$ and $P_r = R_p U_{p1} \Sigma_{p1}^{-1/2}$.

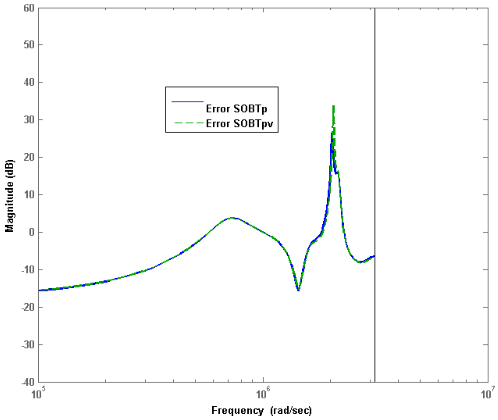


FIGURE 3: Error plot for example 1

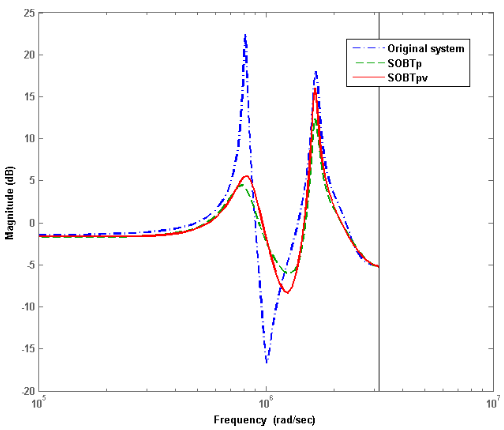


FIGURE 4: ROM response along with actual systemresponse, example 2

*Algorithm 2: Discrete time Position-Velocity Balanced Truncation (SOBTpv)*

*Input:* Given a discrete time stable SOS $H(z)$

*Output:* The ROM $H_r(z)$

1. Calculate the Gramians $W_{pc}$ *and* $W_{vo}$ using (7) and (8).

2. Calculate the Cholesky factors $R_p$ *and* $L_v$ of Gramians of step 1.

3. Compute SVD for the products:

$$R_p^T M^T L_v = [U_{pv1} \quad U_{pv2}] \begin{bmatrix} \Sigma_{pv1} & 0 \\ 0 & \Sigma_{pv2} \end{bmatrix} [V_{pv1} \quad V_{pv2}]^T$$

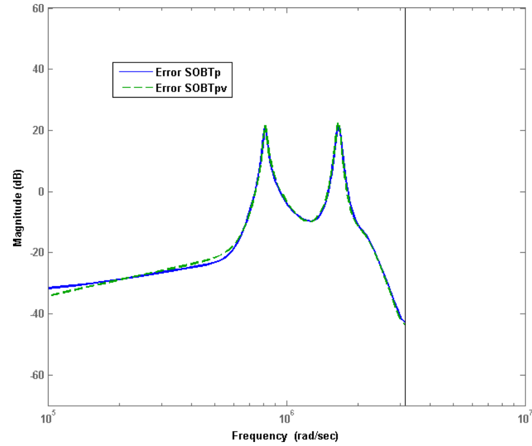


FIGURE 5: Reduction error plot for example 2

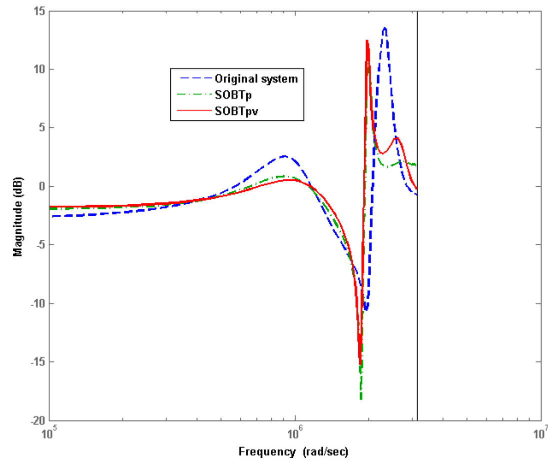


FIGURE 6: ROM response along with actual systemresponse, example 3

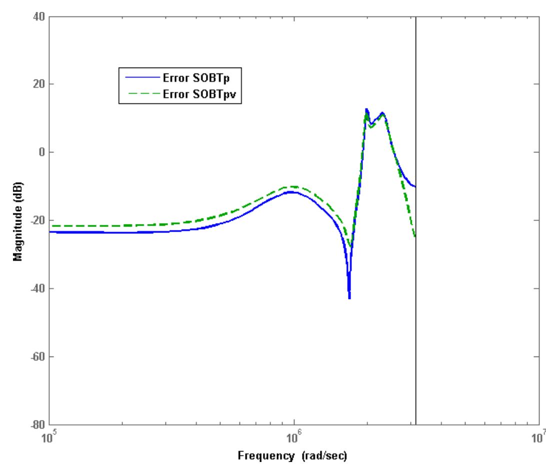


FIGURE 7: Reduction error plot, example 3

where the matrices $[U_{pv1} \quad U_{pv2}]$ and $[V_{pv1} \quad Vp_{v2}]$ are orthogonal and $\Sigma_{pv1} = diag(\xi_1^{pv}, \dots, \xi_r^{pv}), \Sigma_{pv2} = diag(\xi_{r+1}^{pv}, \dots, \xi_n^{pv})$.

4. Calculate the ROM:

$M_r = I_r,\ D_r = D_l^T D P_r,$
$$K_r = P_l^T K P_r, B_{2r} = P_l^T B_2, C_{1r} = C_1 P_r, \quad C_{2r} = C_2 P_r$$
where $P_l = L_v V_{pv1} \Sigma_{\text{pv1}}^{-1/2}$ and $P_r = R_p U_{pv1} \Sigma_{\text{pv1}}^{-1/2}$.

## IV. ILLUSTRATIVE EXAMPLES

Suggested contrivance is solicited to discrete time SOSSs provided in Appendix.

A $6^{th}$ order stable discrete time SOSS of example 1 is curtailed to *r=2* and bode riposte of original as well as resultant ROM is depicted in Figure 1 and the error plot for SOBTp and SOBTpv contrivances is shown in Figure 2.

An $8^{th}$ order stable discrete time SOSS model of example 2 is reduced to *r=6* using suggested contrivances. Bode-riposte for original and ROM is portrayed in Figure 3 and commensurating error plot is portrayed in Figure 4. Finally, a stable $6^{th}$ order discrete time SOSS model of example 3is curtailed to *r=4* and bode-ripost for original and resultingROM is portrayed in Figure 5 and commensurating error plot is depicted in Figure 6. The figures show that the surmised ROM response closely follow the original LSS response forall the example models that ratify the veracious developmentof suggested contrivances.

## V. CONCLUSION

In this manuscript, structure retaining balanced curtailment based MOR framework for discrete time SOSs is suggested. Discrete SOS is supplicated in discrete time generalized algebraic lyapunov equations to procure discretetime position-velocity (interchangably) controlability and observability gramians. formulated gramians are balanced with disparate coalescences to procure sundry balanced transformations. The balanced transformation relents HSVs that provide the basis for balanced curtailment to procure ROM. Suggested contrivances are experimented on disparateSOSSs that ratify the veracious development of the suggested contrivances.

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## APPENDIX

*Example 1:* An 8th order stable discrete time SOS:

$$K = \begin{bmatrix} 1.00 & -.03 & -.04 & 0.15 & 0.13 & -1.0 & 0.00 & 0.10 \\ 0.66 & 0.09 & 0.10 & 0.07 & 0.01 & 0.10 & 1.00 & -1.0 \\ 0.02 & 0.05 & 0.02 & 0.10 & 0.08 & 0.22 & 0.01 & 0.22 \\ 0.09 & 0.10 & 0.01 & 0.03 & 0.12 & 0.10 & 0.12 & 0.10 \\ 0.10 & 0.00 & 0.20 & 0.03 & 0.22 & 0.12 & 0.31 & 0.33 \\ 0.30 & 0.20 & 0.01 & 0.12 & 0.32 & 0.10 & 0.01 & 0.10 \\ 0.01 & 0.10 & 0.02 & 0.04 & -0.0 & 0.11 & 0.00 & 0.00 \\ 0.02 & 0.31 & 0.09 & 0.00 & 0.01 & 0.00 & 0.03 & 0.01 \end{bmatrix}$$

$$D = \begin{bmatrix} 0.20 & 0.03 & 0.01 & 0.38 & 0.20 & -1.0 & 0.10 & 0.00 \\ 0.07 & 0.19 & 0.02 & 0.05 & 0.06 & 0.01 & 0.07 & 0.00 \\ 0.33 & 0.00 & 0.01 & 0.10 & 0.31 & 0.05 & 0.02 & 0.30 \\ 0.12 & 1.00 & 0.10 & 0.14 & 0.16 & 0.01 & 0.00 & 0.20 \\ 0.09 & 0.75 & 0.03 & 0.15 & 0.33 & 0.01 & 0.16 & 0.04 \\ 0.13 & 0.12 & 0.25 & 0.01 & 0.18 & 0.14 & 0.00 & 0.00 \\ 0.15 & 0.01 & 0.52 & 0.12 & 0.01 & 0.23 & 0.01 & 0.02 \\ 0.10 & 0.03 & 0.10 & 0.20 & 0.01 & 0.20 & 0.14 & 0.02 \end{bmatrix}$$

$B_2 = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T$
$C_1 = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$
$C_2 = [-1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0], M = -I_8$

*Example 2:* A stable $6^{th}$ order discrete time SOS:

$$K = \begin{bmatrix} 0.010 & 0.020 & 0.070 & 0.100 & -1.00 & 0.900 \\ 0.600 & 0.200 & 0.200 & 0.100 & -.100 & 0.100 \\ 0.080 & 0.200 & 0.020 & 0.300 & 0.010 & 0.100 \\ 0.001 & 0.400 & 1.000 & 0.200 & 0.400 & 0.070 \\ 0.500 & 0.000 & 0.010 & 0.200 & 0.350 & 0.220 \\ 0.036 & 0.090 & 0.500 & 0.100 & 0.020 & 0.120 \end{bmatrix}$$

$$D = \begin{bmatrix} 0.212 & 0.020 & 0.800 & 0.100 & 0.180 & 0.000 \\ 0.250 & 0.050 & 0.065 & 0.001 & 0.470 & 0.098 \\ 0.200 & 0.080 & 0.040 & 0.098 & 0.000 & 0.450 \\ 0.010 & 0.350 & 0.090 & 0.240 & 0.350 & 0.050 \\ 0.187 & 0.350 & 0.010 & 0.120 & 0.055 & 0.200 \\ 0.180 & 0.065 & 0.032 & 0.212 & 0.082 & 0.420 \end{bmatrix},$$

$B_2 = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T$
$C_1 = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$
$C_2 = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0], M = I_6$

*Example 3:* A 6th order stable discrete time SOS:

$$K = \begin{bmatrix} 1.000 & -.780 & -.030 & 0.050 & 0.500 & -1.00 \\ 0.670 & 0.100 & 0.100 & 0.098 & 0.045 & 0.100 \\ 0.045 & 0.066 & 0.060 & 0.033 & 0.100 & 0.011 \\ 0.040 & 0.200 & 0.010 & 0.120 & 0.240 & 0.030 \\ 0.250 & 0.250 & 0.020 & 0.420 & 0.035 & 0.120 \\ 0.030 & 0.120 & 0.230 & 0.010 & 0.220 & 0.320 \end{bmatrix}$$

$$D = \begin{bmatrix} 0.012 & 0.320 & 0.120 & 1.000 & 0.020 & -1.00 \\ 0.020 & 0.066 & 0.210 & 0.250 & 0.077 & 0.100 \\ 0.120 & 0.001 & 0.220 & 0.010 & 0.031 & 0.050 \\ 0.800 & 0.100 & 0.010 & 0.033 & 0.066 & 0.010 \\ 0.000 & 0.500 & 0.040 & 0.020 & 0.033 & 0.130 \\ 0.050 & 0.022 & 0.055 & 0.020 & 0.182 & 0.142 \end{bmatrix}$$

$B_2 = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T$
$C_1 = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$
$C_2 = [-1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0], M = I_6$