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Balanced Truncation of Unstable Second-Orderly Structured Systems

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Abstract- In study of physical system where suitable surrogate mathematical model need to be derived, it is essential to devise contrivances so that the unstable system behavior could be estimated. To cause unstable second-orderly structured systems (SOSSs) to relent stable reduced order model (ROM) for infinite frequency range, an endeavour for model order reduction is suggested. Second-orderly structure retention is acquired for generalized representation of SOSS. Firstly, the Bernoulli stabilizing solution for unstable SOSS is formulated and supplicated in generalized continuous time algebraic Lyapunov equations (CALEs). The observability and controllability gramians are procured through solution of CALEs, which infact is made possible by stabilizing solution. Gramians are apportioned into velocity and position snippets to achieve the structure retention. Secondly, the equating of formulated position and velocity controllability and observability gramians is performed with disparate coalescences interchangably to generate Hankel singular values (HSVs) for either velocity or position individually or both Velocity and position simultaneously. HSVs represent the extant of engagement of states in system dynamics. Least significant (unimportant) states are curtailed to procure stable ROMs for unstable original SOSSs. Suggested contrivance is assessed on sundry unstable SOSSs. The results ascertain the successful development of the suggested contrivances.

Keywords- Second-orderly systems, model order reduction, infinite gramians, unstable systems, Hankel singular values.

I. INTRODUCTION

System analysts and designers often come across complex and large-scale second-orderly structured systems (SOSSs).The quest for reduced order models (ROM) of these mathematical models arose because of complexity of these models. ROM is a surrogate model of the pristine model. Second-orderly structured systems hold pair of derivatives (1st and 2nd) of the states in system-dynamics. These systems are found in multifarious fields related to engineering and technology like biological systems, electro mechanical technology, image processing, community interaction, smart grid systems, huge-buildings, [1–6] etc. The linear time invariant second-orderly structured system (SOSS) is represented as (1).

$$M\ddot{x}(t) + \Delta\dot{x}(t) + Kx(t) = \beta_2 u(t) \quad (1)$$
$$C_2\dot{x}(t) + C_1 x(t) = y(t)$$

where $M \in R^{nxn}, \Delta \in R^{nxn}, K \in R^{nxn}, \beta_2 \in R^{nxn}, C_2 \text{ and } C_1 \in R^{pxn}, x(t) \in R^n, u(t) \in R^m, y(t) \in R^p$
n = system order; m = number of input(s).
p =number of output(s) for the system.
For (1), number of state-equations is mammoth (in billions) and it becomes rather sturdy or even improbable to handle system complexity for either system analysis or design of the controller in overall curtailment mechanism. This necessitates the quest for model order curtailment mechanism that surrogate large models with lower order identical models called ROMs. ROM form of system depicted in (1) may be described by (2).

$$M_r\ddot{x}_r(t) + \Delta_r\dot{x}_r(t) + K_r x_r(t) = \beta_{2r} u(t) \quad (2)$$
$$C_{2r}\dot{x}_r(t) + C_{1r} x_r(t) = y_r(t)$$

where $M_r \in R^{nxn}, \Delta_r \in R^{nxn}, K_r \in R^{nxn}, \beta_{2r} \in R^{nxm}, C_{2r} \text{ and } C_{1r} \in R^{pxn}, x_r(t) \in R^n, u(t) \in R^m, y_r(t) \in R^p$ and r << n: The augmented generalized first order state space model of (1) may be reiterated as (3).

$$\xi\dot{q} = \Phi q(t) + Bu(t) \quad (3)$$
$$y(t) = Cq(t)$$

With $q(t) = [x(t)^T \dot{x}(t)^T], \xi = \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}, \Phi = \begin{bmatrix} 0 & I \\ -K & -\Delta \end{bmatrix}, B = \begin{bmatrix} 0 \\ \beta_2 \end{bmatrix}, C = [C_1 C_2]$. The ROM of equation (3) in the generalized first order form emerges out as in equation (4).

$$\xi_r \dot{q}_r = \Phi_r q_r(t) + B_r u(t) \quad (4)$$
$$y_r(t) = C_r q_r(t)$$

Where $\xi_r W_r^T \xi T_r$, $\Phi_r = W_r^T \Phi T_r$, $B_r = W_r^T B$, $C_r = CT_r$, $W_r \; and \; T_r$ are called balanced transformation matrices. But these transformation matrices are not computable for standard unstable systems [7]. The solution of generalized structure continuous time algebraic Lyapunov equations (CALEs) of (5) and (6), provides controllability and observability gramians for curtailment.

$$\xi^T G_o \Phi + \Phi^T G_o \xi = -C^T C \quad (5)$$
$$\xi G_c \Phi^T + \Phi G_c \xi^T = -BB^T \quad (6)$$

Where Pc and Qo are controllability and observability gramians for system (3) in the stated order.
When curtailment takes place, ROM in (4) must not show haphazard pattern of entries of matrices of (1) i.e. μ, η, K.

II. RELATED WORK

First method for model order reduction (MOR) was initiated by [8] in 1966 for generalized systems which was further modified by [9-10]. Structure-retaining extensions of classical model curtailment methods such as dominant pole contrivance, modal truncation [11], moment-matching [12], balanced truncation [13], or for example of the H2-optimal iterative rational Krylov contrivance [14]. In [15], author introduced an equating concept that is strongly related to the origins of balanced curtailment than the other extensions in this regard. In the context of 1st order systems, contrivances in [16], are aimed at such local approximations. Nevertheless, earlier curtailment approaches did not retain the structure in ROM [17-18]. But later developed MOR contrivances like modified Arnoldi, second-orderly structured balanced truncation method (SOBTM) and moment-matching (based on Krylov-subspaces) [2], [12], [15], [18–22] etc. retained second-orderly structure in ROM. In Krylov, Arnoldi, or moment-matching contrivances, although, curtailment of the model was achieved but ROM got unstable in the due course. Accordingly, a-priori curtailment error bounds do not exist. But stability of ROM is retained and so exist the a-priori curtailment error-bound for SOBTM [17-18], [23–25]. [13] confabulated multifarious SOBTM contrivances for stable SOSSs. It looks reasonable as long as the system is stable. The instability of the system poses a serious threat to the reduction mechanism as the CALEs (5) and (6) get unresolvable and the arithmetic of controllability and observability gramians using CALEs is halted. Numerous MOR contrivances [7], [26–34] etc. in this direction for sundry unstable systems have been endorsed. Once these contrivances stabilize the unstable system in a prior step, the curtailment mechanism is endorsed. But irony of the situation is that, these contrivances do not bother about the issues faced when MOR of unstable SOSSs takes place. None of the aforecited contrivances contribute to second orderly structured forms of pristine unstable systems. Provision of structure-retention or physical exposition retention in ROM is not addressed at all. Eventually ROM generated through this stratagem does not allow analysis and design mechanism to be conducted and ROM becomes meaningless. The aforesaid contrivances are not solicitable for MOR applications of unstable large scale SOSSs and, no MOR stratagem for curtailment of large scale unstable SOSS exists in literature. Keeping in view the said research gap, in present work a new MOR contrivance for MOR of unstable SOSSs is suggested. The suggested contrivance conserve the second order form along with provision of stable ROMs. In the suggested work here, firstly, the LSS in equation (3) is supplicated in Bernoulli stabilizing equations that relent the stability feedback matrices to procure stable closed form of original system. The CALEs get solvable when feedback-stabilized LSS is supplicated to relent controllability and observability gramians. For SOSS structure retention in equation (3), computed gramians are partitioned into velocity and position snippets of respective controllability and observability gramians to solicit balanced truncation. Secondly, disparate balanced transformations are procured by equating position and velocity gramians with disparate coalescences to procure position, velocity, position-velocity, velocity-position Hankel singular values (HSVs) that represent importance of respective states in system dynamics. Least important states are curtailed to procure ROMs. The suggested algorithms for position and position- velocity equating of SOSSs are assessed on sundry unstable systems for certification of suggested development. Results for few simulatory systems and one benchmark example from [35] are presented and performance juxtaposition of suggested contrivances is discussed. The organization of the paper goes as follows: next presents preliminary discussion on SOSSs followed by the suggested contrivances in section 4. In section 5, results and discussion are presented and paper is concluded.

III. SECOND-ORDERLY STRUCTED SYSTEMS

Transfer function of the second-orderly system of equation (1) can be shown as

$$H(s) = (sC_2 + C_1)(s^2M + s\Delta + K)^{-1}\beta_2 \quad (7)$$

and its brief representation is written as $H = [M, \Delta, K, \beta_2, C_1, C_2]$. System (1) is stable if all the zeros of P (λ) = λ2M + λD +K, have negative real parts (i.e lie in the left half plane) or unstable if any of zeros of P (λ) = λ2M + λD + K, s have positive real parts (i.e lie in the right half plane). The conditions of observability as well as controllability for system (1) and (3) are described in [13]. System (1) can be transformed into an equivalent form by employing a system equivalence transformation (Tl, Tr) with Tl and Tr kept as nonsingular as given in (8).

$$\bar{M} = P_l M P_r \quad \bar{\Delta} = P_l \Delta P_r \quad \bar{K} = P_l K P_r \quad (8)$$
$$\bar{\beta}_2 = P_l \beta_2 \quad \bar{C}_1 = C_1 P_r \quad \bar{C}_2 = C_2 P_r$$

Also corresponding first order transformed form of (3) becomes:

$$\xi = \bar{P}_l \xi \bar{P}_r \quad \bar{\Phi} = \bar{P}_l \Phi \bar{P}_r \quad \bar{B} = \bar{P}_l B \quad \bar{C} = \bar{P}_r C$$

Where

$$\bar{P}_l = \begin{bmatrix} P^{-1} & 0 \\ 0 & P_l \end{bmatrix}, \qquad \bar{P}_r = \begin{bmatrix} P_r & 0 \\ 0 & P_r \end{bmatrix}$$

If we consider system (1) as stable then observability and controllability gramians for system (3) given by:

$$(\Phi - BK_c)G_{cstab}\xi^T + \xi G_{cstab}(\Phi - BK_c)^T = -BB^T \quad (9)$$

$$(\Phi - K_oC)^T G_{ostab}\xi + \xi^T G_{ostab}(\Phi - K_oC) = -C^TC \quad (10)$$

are the positive semi-definite, symmetric and unique solutions of the generalized Lyapunov equations (6) and (5). Where $\varphi_S(t) = e^{E^{-1}At}E^{-1}$ is the fundamental solution matrix of equation (3). The gramians of equations (9) and (10) can be partitioned as

$$G_{cstab} = \begin{bmatrix} G_{pcstab} & G_{12cstab} \\ G_{12cstab}^T & G_{vcstab} \end{bmatrix}, G_{ostab} = \begin{bmatrix} G_{postab} & G_{12ostab} \\ G_{12ostab}^T & G_{vostab} \end{bmatrix}$$

Where $G_{vostab}, G_{postab}, G_{vcstab}$ and G_{pcstab} are velocity and position observability and controllability gramians respectively.

Remark: The partitioning of gramians in position and velocity snippets ensure the SOSS structure retention in ROMs. The velocity and position gramians can be transformed into:

$$\bar{G}_{pc} = \bar{P}_r^{-1} G_{pc} \bar{P}_r^{-T} \qquad \bar{G}_{vc} = \bar{P}_r^{-1} G_{vc} \bar{P}_r^{-T}$$
$$\bar{G}_{po} = \bar{P}_r^{-T} G_{po} \bar{P}_r \qquad \bar{G}_{vc} = \bar{P}_r^{-T} G_{vo} \bar{P}_l^{-1}$$

by soliciting system equivalence transformation (Pl, Pr). As a result, product of the factors given above relents following definition.

Definition 1: Consider a stable system as in equation (1)[13]. SQURT of eigenvalues of product

1. The square root of the eigenvalues of product G_{pcstab}, G_{postab} are the position HSVs.
2. The square root of eigenvalues of product $G_{vcstab}M^TG_{vostab}M$ are the velocity HSVs.
3. The square root of eigenvalues of product $G_{pcstab}M^TG_{vostab}M$ are the position-velocity HSVs.
4. The square root of eigenvalues of product $G_{vcstab}G_{postab}$ are the velocity position HSVs.

The resultant ROM for SOSS in (1) is acquired by soliciting balanced truncation which is based on the magnitudes of HSVs. In case when SOSS of equation (1) is originally un- stable, the solution of Lyapunov eqn. (6) and (5) becomes irrealizable (indefinite) and hence halts the mechanism of balanced truncation. To avert this constraint, stabilization solution is suggested in the next section.

IV. SUGGESTED METHODOLOGY

Bernoulli stabilization solution presented below is computed and supplicated in CALEs to procure gramians for equating. This makes CALEs (6) and (5) solvable for unstable systems

A. Stabilizing Solution for Unstable SOS

To stabilize system (3), Bernoulli feedback solution [21] and gramians are computed from (16)-(19)

$$(\Phi - BK_c)G_{cstab}\xi^T + \xi G_{cstab}(\Phi - BK_c)^T = -BB^T \quad (12)$$

$$(\Phi - K_oC)^T G_{ostab}\xi + \xi^T G_{ostab}(\Phi - K_oC) = -C^TC \quad (13)$$

where $Kc = B^TXc\,\xi$ and $Ko = \xi\, XoC^T$ are known as Bernoulli stabilizing feedback matrices, since the matrices Xc and Xo are the respective stabilizing solutions of the generalized algebraic Bernoulli equations (18) and (19)

$$\xi^T X_c \Phi + \Phi^T X_c \xi = \Phi^T X_c B B^T X_c \xi \quad (14)$$
$$\Phi X_o \xi^T + \Phi X_o A^T = \xi X_o C^T C X_o \xi^T \quad (15)$$

Partitioning of gramians obtained from (12) and (13) yields

$$G_{cstab} = \begin{bmatrix} G_{pcstab} & G_{12cstab} \\ G_{12cstab}^T & G_{vcstab} \end{bmatrix},$$
$$G_{ostab} = \begin{bmatrix} G_{postab} & G_{12ostab} \\ G_{12ostab}^T & G_{vostab} \end{bmatrix}$$

Where $G_{vostab}, G_{postab}, G_{vcstab}$ and G_{pcstab} are velocity and position observability and controllability gramians respectively.

Remark: The gramians are computed from (16) and (17) using stabilized system in proposed technique-II. Moreover, partitioning of gramians in to position and velocity portions, ensures the structure preservation in ROM. Also, in proposed technique-II, there is no need to augment any large order unstable portion.

Consideration of these aspects, ensures a better and meaningful approximation of original system:

$\bar{G}_{pc} = \bar{P}_r^{-1} G_{pc} \bar{P}_r^{-T}$ $\bar{G}_{vc} = \bar{P}_r^{-1} G_{vc} \bar{P}_r^{-T}$

$\bar{G}_{po} = \bar{P}_r^{-T} G_{po} \bar{P}_r$ $\bar{G}_{vc} = \bar{P}_r^{-T} G_{vo} \bar{P}_l^{-1}$

The product of position and velocity gramians yields following definition.

Definition 1: Consider a stable system as in equation (1)[11].

1. The square root of the eigenvalues of product G_{pcstab}, G_{postab} are the position HSVs.
2. The square root of eigenvalues of product $G_{vcstab} \mathrm{M}^{\mathrm{T}} G_{vostab} \mathrm{M}$ are the velocity HSVs.
3. The square root of eigenvalues of product $G_{pcstab} \mathrm{M}^{\mathrm{T}} G_{vostab} \mathrm{M}$ are the position-velocity HSVs.
4. The square root of eigenvalues of product $G_{vcstab} G_{postab}$ are the velocity position HSVs.

Balanced Transformation for Stabilized SOS

For second order balanced transformation, different balancing techniques can be applied to stabilized system in order to obtain velocity and position HSVs.

Definition 2: The stabilized SOS is called:

1. Position balanced if $G_{pcstab} = G_{postab} = diag(\zeta_1^{pstab}, \dots, \zeta_n^{pstab})$.
2. Velocity balanced if $G_{vcstab} = G_{vostab} = diag(\zeta_1^{vstab}, \dots, \zeta_n^{vstab})$.
3. Position-velocity balanced if $G_{pcstab} = G_{vostab} = diag(\zeta_1^{pvstab}, \dots, \zeta_n^{pvstab})$.
4. Velocity-position balanced if $G_{vcstab} = G_{postab} = diag(\zeta_1^{vpstab}, \dots, \zeta_n^{vpstab})$.

Where ζ represents either velocity or position or both velocity-position (position-velocity) HSVs arranged in descending order.

Balanced transformation can be derived by considering the Cholesky factorization of gramians given as:

$$G_{pcstab} = R_{pstab} R_{pstab}^T \quad G_{vcstab} = R_{vstab} R_{vstab}^T$$
$$G_{postab} = L_{pstab} L_{pstab}^T \quad G_{vostab} = L_{vstab} L_{vstab}^T \tag{16}$$

where $R_{pstab}, R_{vstab}, L_{pstab}$ *and* $L_{pstab} \in R^{nxn}$ are lower triangular non-singular Cholesky factors which are used to find out HSVs via classical singular values as given below.

$$(\zeta_i^{pstab})^2 = \lambda_i(G_{pcstab} G_{postab}) = \lambda_i(G_{pstab} R_{pstab}^T L_{pstab} L_{pstab}^T)$$
$$= \lambda_i(L_{pstab}^T R_{pstab} R_{pstab}^T L_{pstab}) = \sigma_i^2(R_{pstab} L_{pstab})$$

Where $\sigma_{istab}{}'s$ are classical SV's. Similarly, $\zeta^{pstab} = \sigma_{istab}(R_{vstab}^T M^T L_{vstab})$, $\zeta_i^{pvstab} = \sigma_{istab}(R_{pstab}^T M^T L_{vstab})$ and $\zeta^{vpstab} = \sigma_{istab}(R_{vstab}^T M^T L_{pstab})$Calculating the singular value decomposition (SVD) of these products provide.

$$R_{pstab}^T L_{pstab} = U_{pstab} \Sigma_{pstab} V_{pstab}^T$$
$$R_{vstab}^T M^T L_{vstab} = U_{vstab} \Sigma_{vstab} V_{vstab}^T$$
$$R_{pstab}^T M^T L_{vstab} = U_{pvstab} \Sigma_{pvstab} V_{pvstab}^T$$
$$R_{vstab}^T L_{pstab} = U_{vpstab} \Sigma_{vpstab} V_{vpstab}^T \tag{17}$$

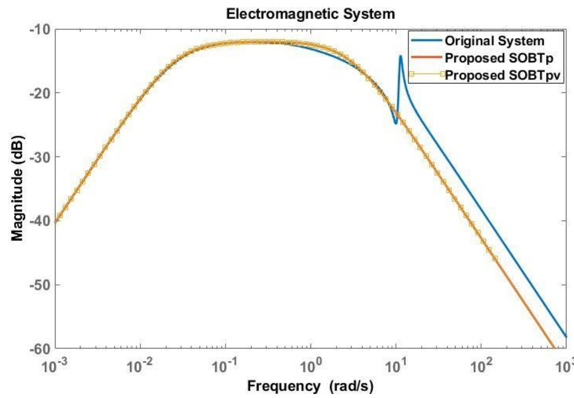

FIGURE 1: Bode plots for Electromagnetic System

where $U_{pstab}, V_{pstab}, U_{vstab}, V_{vstab}, U_{pvstab}, V_{pvstab}, U_{vpstab}, V_{vpstab}$ are the orthogonal matrices while

$\Sigma_{pstab} = diag(\zeta_1^{pstab}, \dots, \zeta_n^{pstab})$ $\Sigma_{vstab} = diag(\zeta_1^{vstab}, \dots, \zeta_n^{vstab})$

$\Sigma_{pvstab} = diag(\zeta_1^{pvstab}, \dots, \zeta_n^{pvstab})$ $\Sigma_{vpstab} = diag(\zeta_1^{vpstab}, \dots, \zeta_n^{vpstab})$

are non-singular matrices. Relationship in equation (17) can be utilized to find out the balanced transformation ($P_{lstab} P_{rstab}$) for techniques of definition 2.

Balanced Truncation for Stabilized SOS

As mentioned in the previous section, the balanced transformation yields HSVs. The magnitudes of these HSVs show the extent to which the velocity and position states are involved in the dynamics of the system. States corresponding

to small magnitudes have minute involvement and are truncated at small cost of reduction error. Two algorithms for velocity and position balanced truncation are presented here for stabilized SOSs. In algorithm 1, the transformation matrix P_{lstab} is chosen in such a way that

$$G_{pcstab} = G_{postab} = G_{vostab} = diag(\zeta_1^{pstab}, \dots, \zeta_r^{pstab})$$

and in algorithm 2 position controllability gramian is balanced with velocity observability gramian.

Algorithm 1: Position Balancing for Stabilized SOS (SOBTp)

Input: Given a stable large-scale SOS $H = [M, \Delta, K, \beta_2, C_1, C_2]$

Output: The ROM $H_r = [M_r, \Delta_r, K_{r,}, \beta_{2r}, C_{1r}, C_{2r}]$

1. Calculate the gramians $G_{pcstab}, G_{postab}, G_{vcstab}\ and\ G_{vostab}$ using equations (12) and (13).
2. Calculate the Cholesky factors $R_{pstab}, R_{vstab}, L_{pstab}\ and\ L_{vstab}$ of Gramians using equation (16).
3. Compute SVD for the products:

$$R_{pstab}^{T} L_{pstab} = [U_{p1stab}\ \ U_{p2stab}] \begin{bmatrix} \boldsymbol{\Sigma_{p1stab}} & 0 \\ 0 & \boldsymbol{\Sigma_{p2stab}} \end{bmatrix} [V_{p1stab}\ \ V_{p2stab}]^{T}$$

$$R_{vstab}^{T}\, M^{T}\, L_{vstab} = [U_{v1stab}\ \ U_{v2stab}] \begin{bmatrix} \boldsymbol{\Sigma_{v1stab}} & 0 \\ 0 & \boldsymbol{\Sigma_{v2stab}} \end{bmatrix} [V_{v1stab}\ \ V_{v2stab}]^{T}$$

Where the matrices are $[U_{p1stab}\ \ U_{p2stab}]$, $[V_{p1stab}\ \ V_{p2stab}]$, $[U_{v1stab}\ \ U_{v2stab}]$ and $[V_{v1stab}\ \ V_{v2stab}]$ are orthogonal and

$$\Sigma_{p1\mathrm{stab}} = diag(\zeta_1^{pstab}, \ldots, \zeta_r^{pstab}), \quad \Sigma_{p2\mathrm{stab}} = diag(\zeta_{r+1}^{pstab}, \ldots, \zeta_n^{pstab})$$

$$\Sigma_{v1\mathrm{stab}} = diag(\zeta_1^{vstab}, \ldots, \zeta_r^{vstab}), \quad \Sigma_{v2\mathrm{stab}} = diag(\zeta_{r+1}^{vstab}, \ldots, \zeta_n^{vstab})$$

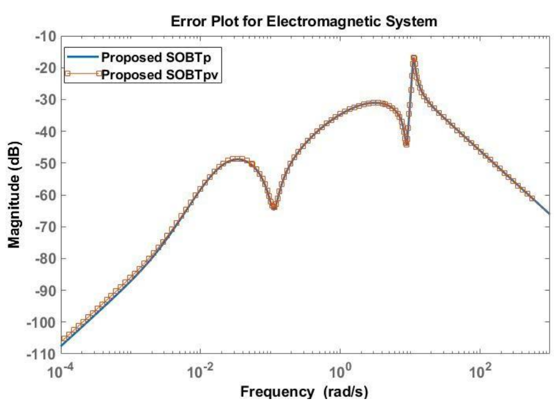

FIGURE 2: Error-plots for Electromagnetic System

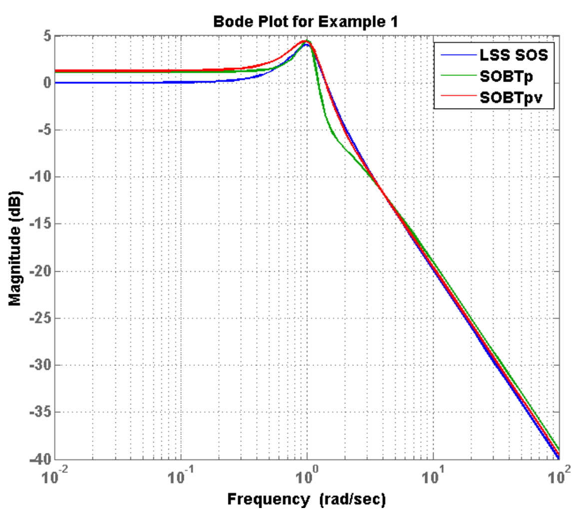

FIGURE 3: Bode plots for Example 1

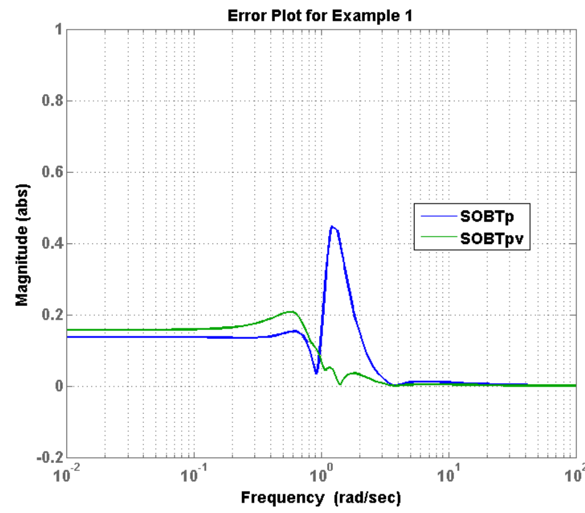

FIGURE 4: Error-plots for Example 1

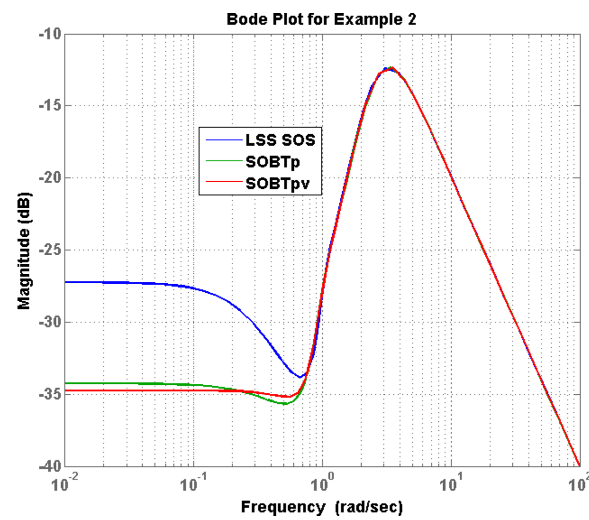

FIGURE 5: Bode plots for Example 2

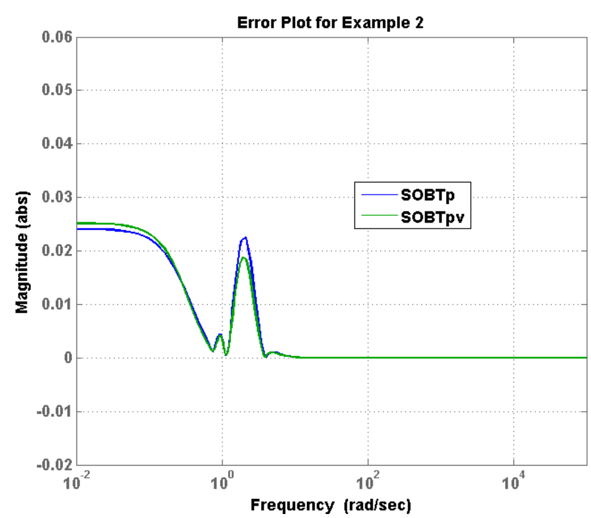

FIGURE 6: Error-plots for Example 2

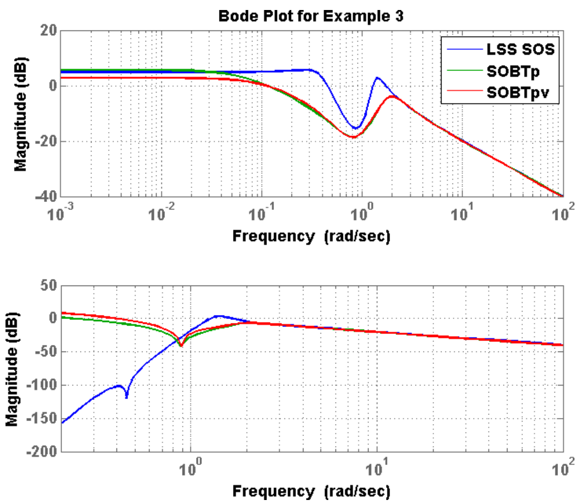

FIGURE 7: Bode plots for Example 3

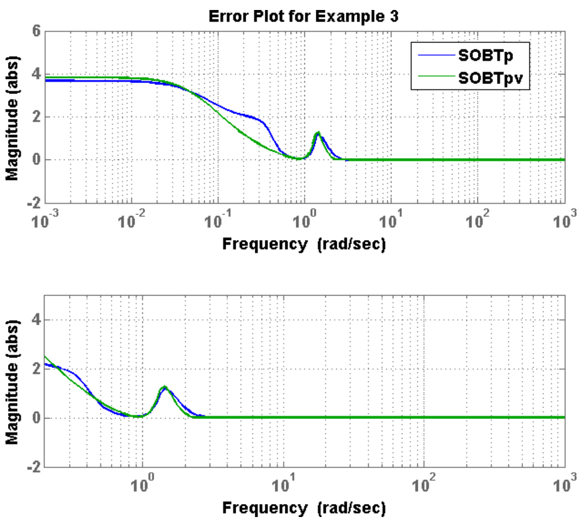

FIGURE 8: Error-plots for Example 3

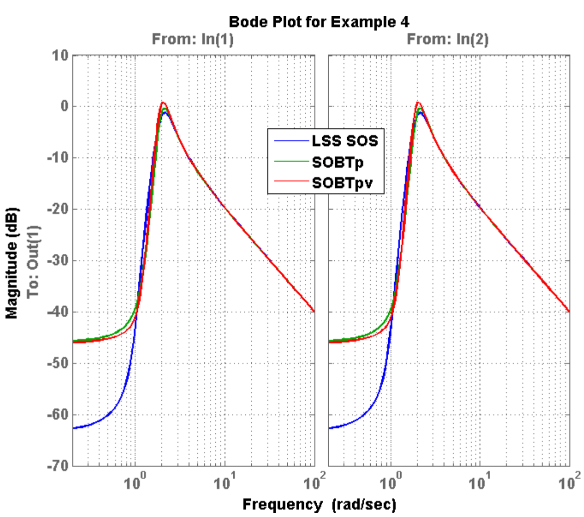

FIGURE 9: Bode plots for Example 4

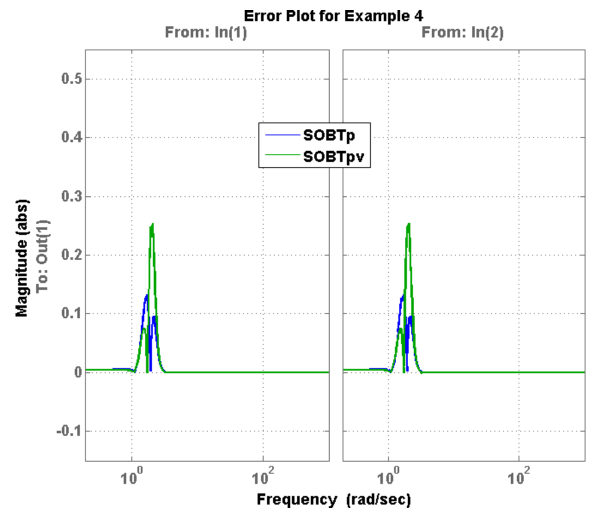

FIGURE 10: Error-plots for Example 4

4. Calculate the ROM

$M_r = P_{lstab}^T M P_{rstab}$ $\quad D_r = P_{lstab}^T D P_{rstab}$

$K_r = P_{lstab}^T K P_{rstab}$

$B_{2r} = P_{lstab}^T B_2$ $\quad C_{1r} = C_1 P_{rstab}$

$C_{2r} = C_2 P_{rstab}$

Where $P_{lstab} =$

$L_{vstab}\ V_{v1stab}\ \Sigma_{plstab}^{-1/2}$ *and* $P_{rstab} =$

$R_{pstab}\ U_{P1stab}\ \Sigma_{plstab}^{-1/2}$

Algorithm 2: Position-Velocity Balancing for Stabilized SOS(SOBTpv)

Input: Given a stable large-scale SOS $H = [M, \Delta, K, \beta_2, C_1, C_2]$

Output: The ROM $Hr = [Mr, \Delta r, Kr, \beta_{2r}, C_{1r}, C_{2r}]$

1. Calculate the gramians G_{pcstab} and G_{vostab} using equations (16) and (17).

2. Calculate the Cholesky factors R_{pstab} and L_{vstab} of gramians

G_{pcstab} and G_{vostab} as given in equation (20).

3. Compute SVD for the products:

$$R_{vstab}^T M^T L_{vstab}$$
$$= [U_{pv1stab}\ U_{pv2stab}] \begin{bmatrix} \Sigma_{pv1stab} & 0 \\ 0 & \Sigma_{pv2stab} \end{bmatrix} [V_{pv1stab}\ V_{pv2stab}]^T$$

Where

$[U_{pv1stab}\ U_{pv2stab}]$ *and* $[V_{pv1stab}\ V_{pv2stab}]$ are orthogonal and

$\Sigma_{pv1stab} = diag(\zeta_1^{pvstab}, \dots, \zeta_r^{pvstab})$

$\Sigma_{pv2stab} = diag(\zeta_{r+1}^{pvstab}, \dots, \zeta_n^{pvstab})$

4. Calculate the ROM

$M_r = I_n$ $\quad D_r = P_{lstab}^T D P_{rstab}$

$K_r = P_{lstab}^T K P_{rstab}$

$B_{2r} = P_{lstab}^T B_2$ $\quad C_{1r} = C_1 P_{rstab}$

$C_{2r} = C_2 P_{rstab}$

Where $P_{lstab} =$

$L_{vstab}\ V_{pv1stab}\ \Sigma_{pv1stab}^{-1/2}$ *and* $P_{rstab} =$

$R_{pstab}\ U_{Pv1stab}\ \Sigma_{pv1stab}^{-1/2}$

Remark: As the gramians obtained from equations (12) and (13) are symmetric, unique and positive definite solutions of CALEs, therefore the obtained ROMs are stable.

V. RESULTS AND DISCUSSION

The suggested MOR contrivances are solicited to sundry unstable SOSSs like electromagnetic system and a few self- generated examples as mentioned in appendix and results are portrayed.

A. Electromagnetic System

An electromagnetic energy harvesting system with interconnect-network is deployed between transmission lines as depicted in Figure 1. [35]. Parameters in unit length of the lossy coupled SISO interconnect network at $\Gamma_0 = 25^oC$.

The model (7) is an unstable SOSS system for three stages of RLC network. This system is diagonally extended to twelve stages and solicited the suggested curtailment contrivances. The 12th order system is curtailed to r = 1 using suggested SOBTp and SOBTpv contrivances. The response for ROM over infinite snippet is shown in Figure0. This system is curtailed to r = 3 using algorithm 1 and 2. The bode-magnitude and error-plots for electromagnetic system and ROM with second order balanced truncation position (SOBTp) and second order balanced truncation position velocity (SOBTpv) contrivances is depicted in Figure 1 and 2 respectively. Note that the curtailed model is very closely surmising the original SOSS.

B. Simulation Examples

The proposed MOR techniques are applied to multiple unstable SOSs and results for few examples (given in appendix) are presented.

SISO Systems: The 9th order system in Example 1 is a single input single output (SISO) system. This system is curtailed to r = 3 using algorithms 1 and 2. The bode-magnitude and error-plots for original SOSS of Example 1 and ROM with second order balanced truncation position (SOBTp) and second order balanced truncation position- velocity (SOBTpv) contrivances is depicted in Figure 3 and 4 respectively. Note that the curtailed model is very closely surmising the original SOSS. Further, 11th order SISO SOSS of Example 2 is curtailed to r = 4 and responses for original and curtailed systems are presented in Figure 5 and 6 respectively. It is again observed that ROMs are closely depicting the original SOSS response.

*MIMO Systems***:** Moreover, suggested contrivances are assessed on multi input multi output (MIMO) systems. The 9th order system of Example 3 with 2 inputs 2 outputs is curtailed to r = 3 and response for original and ROMs using SOBTp and SOBTpv contrivances is depicted in Figure 7 and 8 respectively. It is evident in this figure that the ROMs are closely surmising original large order MIMO system of Example 3. Finally the 10th order MIMO system of Example 4 with 2 inputs and 1 output is curtailed to r = 5 and bode- plots for original and ROMs procured using the suggested contrivances are shown in Figure 9 and 10 respectively. These plots also ascertain that the ROMs using suggested contrivances are comprehensively surmising the large order systems.

VI. CONCLUSION

A structure retaining contrivance for model order curtailment of unstable SOSSs that relent stable ROM is suggested. The Bernoulli stabilizing solution for unstable SOSS is formulated and supplicated in CALEs. The solutions allow CALEs to be solved in order to procure the controllability and observability gramians. For structure retention, gramians are partitioned into position and velocity snippets. The position and velocity controllability and observability gramians are balanced with disparate coalescences to procure position and velocity or both HSVs. States with less HSV magnitudes are curtailed to procure stable ROMs for unstable original SOSSs. suggested contrivance is assessed on sundry unstable SOSSs and results for SISO and MIMO systems are presented. The results ascertain the correct development of the suggested contrivance.

VII. ACKNOWLEDGEMENTS

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APPENDIX

$$K=\begin{bmatrix} -1.0000 & -0.4195 & -0.2795 & -1.0115 & -1.0000 & -0.4195 & -0.2795 & -1.0115 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\Delta=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1.7184 & 0.6264 & 0.1954 & 0.3851 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\beta 2=\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}$, M= I9 $\quad C1 =$
$[1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]$, M=I9
$C2=[1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]$, M=I9

Example 2: An 11th order unstable continuous time SISO SOS

$$K=\begin{bmatrix} 0 & -49.5 & -23 & -39 & -16.5 & -49.5 & -23 & -39 & -16.5 & -40.5 & -23 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\Delta=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 154.5 & 69.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$\beta 2=\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}$ $\quad C1 =$
$[1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1] \quad C2 =$
$[1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]$, M = I11

Example 3: A 9th order unstable continuous time MIMO SOS

$$K=\begin{bmatrix} -0.0928 & 0 & -0.0256 & -0.0934 & -0.9466 & -0.039 & -0.0256 & 0 & 0.5731 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\Delta=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0.0053 & 57.5662 & 1.8751 & 0.03581 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\beta 2=\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$,

$C1=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -10 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$,

$C2=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, M=I9

Example 4: A 10th order unstable continuous time MIMO SOS

$$K=\begin{bmatrix} 2 & 2 & 10 & -160 & -20 & -20 & -100 & -240 & -280 & -1440 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\Delta=\begin{bmatrix} 0 & 0 & 0 & -1.96 & 120 & -80 & 60 & 0.2 & 17.6 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\beta 2=\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
$C1=[1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]$
$C2=[1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]$, M = I10