Equated Reduction of Unstable Bounded Frequency Second- Orderly Structured Systems

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Abstract- Physical systems are usually expressed in terms of mathematical models and there is a strong desire to represent these mathematical models with equivalent curtailed models. In this paper, structure retaining second-orderly equated reduction contrivances for second-orderly structured systems (SOSSs) order curtailment with unstable form in bounded frequency hiatus are bestowed. To formulate the gramianns for equating, the continuous time algebraic Lyapunov equations (CALEs) for unstable SOSSs get unresolvable, that often hinders the performance of reduction process. To overcome this restraint, system is stabilized using Bernouli-feedback stabilization process in a foremost step. On the way of performance emphasis of reduced-order model (ROM) in intended bounded hiatus, bounded frequency gramianns for stabilized system are defined and calculated by utilizing the suggested bounded frequency CALEs. The fragmentation of resultant gramianns in the position/velocity sections goes on to retain the structure in ROM so that exposition of pristine system is retained. The position and velocity gramianns are compared in disparate steps to procure sundry second-orderly equating contrivances that relent ROM with optimal performance in desired frequency band. The juxtaposition of proposed procedure with infinite gramianns approaches is conferred for multifarious systems to confirm the accurate improvements and supremacy of proposed methods over existing approaches.

Keywords- Second Order Structured Systems, Model Reduction, Unstable Systems, Bounded Frequency Gramianns, Hankel Singular Values.

I. INTRODUCTION

Mathematical model derivation has drawn focus of the system analysts and designers over the time. Practical systems frequently utilize complex and highly advanced mechanisms for modeling, leading to multidimensional mathematical approaches [1]. Traditional analysis based approaches and control algorithms do not confer effective solutions for these systems with bounded computational power [2]. Reduced dimensions of the subjects of similar kind are desirable to address the matter concerned. Reducedorder contrivances are leaned towards the development of product with larger dimensioned system analysis, synthesis, control design, and simulation because they are implemented quickly, less expensive, and easy to use [3]. The model reduction approaches (MRA) may be computationally efficient, simplistic, and stable and should result in a curtailed order model (COM) that retains specific complex system attributes, such as stability, and passivity in both high and low frequency bands. A second order structured (SOS) system encompasses differential-pairs for first as well as second order states in dynamical system Such dynamical system's approximations are applicable in circuit simulation, huge structures, power system analysis [4], structuro-dynamical acoustics [19], control system theory [5-9] and marine systems [10-15] etc. A second order system (SOS) that is linear and time invariant is portrayed as (1).

$$\mu_s\ddot{x}(t) + \eta_s\dot{x}(t) + \mathrm{K}_s x(t) = \beta s_2 u_s(t) \qquad (1)$$

while $\eta_s \in \Re^{n\times n}, \mu_s \in \Re^{n\times n}, \beta s_2 \in \Re^{n\times m}, \mathbf{K}_s \in \Re^{n\times n} C s_2$ and $C s_1 \in \Re^{p\times n}, x(t) \in \Re^n, y(t) \in \Re^p$ and $u_s(t) \in \Re^m, n$ is sys ordered, m are input(s) number, p are output(s) number of the mentioned system.

BTW in (1), no. of stated equations is mammoth (in billions) and gets even exaggerated and hardy, leading to system complexity for both design and system analysis. Same goes true for the controller. Consequently, we are left with no option but to go for model approximation process in which larger models are surrogated with similar lower order models termed as reduced-order models (ROMs). Reduced and approximated form of (1) may be entailed by (2).

$$\mu_{sr}\ddot{x}_r(t) + \eta_{sr}\dot{x}_r(t) + {}_{rsr}x_r(t) = \beta s_2 r u_s(t)_{Cs_2 r\dot{x}_r(t)+Cs_1 r x_r(t)=y_r(t)} = Cs_{1r} \quad (2)$$

while $\mu_{sr} \in \Re^{r\times r}, \eta_{sr} \in \Re^{r\times r}, \mathbf{K}_{sr} \in \Re^{r\times r}, \beta s_{2r} \in \Re^{r\times m}, Cs_{1r}$ and $Cs_{2r} \in \Re^{p\times r}, x_r(t) \in \Re^{r}, u_s(t) \in \Re^{m}$ and $y_r(t) \in \Re^{p}$, and $r \ll n$. In (3) general structure in the first order spatial state form of (1) is entailed as.

$$\curlyvee\,\dot{q} = \alpha q(t) + Bu_s(t)$$

with $q(t) = [x(t)^T \dot{x}(t)^T], \gamma = \begin{bmatrix} I_s & 0 \\ 0 & \mu_s \end{bmatrix}, \alpha = \begin{bmatrix} 0 & I_s \\ -\mathrm{K}_s & -\eta_s \end{bmatrix}, B = \begin{bmatrix} 0 \\ \beta s_2 \end{bmatrix}, Cs = [Cs_1 \quad Cs_2]$.

And when ROM (2) written in the generalized structural arrangement may be entailed as (4).

$$\curlyvee_r\,\dot{\tilde{q}}_r = \alpha_r \tilde{q}_r(t) + \beta s_r u_s(t) \quad (3)$$

$$y_r(t) = Cs_r \tilde{q}_r(t) \quad (4)$$

Where $\curlyvee_r = W_r^T \curlyvee T_r, \alpha_r = W_r^T \alpha T_r, \beta s_r = W_r^T B, Cs_r = CsT_r . T_r$ and W_r are called equated X-formation matrics. However, these X-formation matrics found not calculatable for accepted unstable sys [16]. The solution of common-structure continual (in time) algebric Lyapnov equations (CALEs) of (5) and (6), gives controlability and obsevability gramianns for approximation.

$$\curlyvee^T Q_{o.}\alpha + \alpha^T Q_{o.}\curlyvee = -Cs^T Cs \quad (5)$$

$$\curlyvee P_{c.}\alpha^T + \alpha P_{c.}\curlyvee^T = -BB^T \quad (6)$$

When approximation process occurs, ROM (4) shows a randomized pattern of elements of (1).

II. RELATED WORK

First MOR method was brought in by [17] in mid sixties for general systems and then further improved by [18-19]. Nevertheless, Moore pioneered the notion of curtailed balancing in early eighties [20]. retaining edifice developments of classic model reduction techniques say for example, dominanted pole procedure, reduced models [21], matched moments [22], equated curtailment [23], or in case of H2-optimum IR Krilov procedure [24]. In the frequency domain, BT is a frequently used model reduction methodology. The basic concept is ignoring the least controllable and observable states having no significant impact on the behavior of system with the higher dimensions. In this approach, the ROM is attained by eliminating the least observable and controllable states of the balanced system [7], [25-26]. Further simplification of this method is explained in [5-6], [8-9]. The balanced realization is broadened for bigger dimension time variated systems. To reduce the large, linear and discrete-time, the frequency-weighed equated realization method and normally equated realization technique of linear time invariated systems are solicited in [37]. The method of balanced realization to simplify complex parameteric-dependence units is provided in [38] and parameters are preserved in ROM. This procedure is then extended to simplify the bigger dimension nonlinear functions directly by formulating the incremented controlability function and incremented obsevability function [39]. While in [40], the nonlinear model reduction process is performed for the power system by application of the empirical controlability covariant and empirical obsevability covariant which are found around the operational points. A detailed view on the model reduction using equated realization technique and frequency-weighed equated diminution are mentioned in [41-42]. A strongly directed to origins of the equated reduction than some other adjuncts, equating concept was introduced by [27]. Aimed at such local approximations in the conditions of 1st order systems, contrivances in [28], are encountered. However, reduction mechanisms such as in [29], [30] did not maintain the ROM structure. But this endeavor was accomplished later by development of MOR approaches e.g. modified Arnoldi, 2nd order structured equated reduction method (SOETM) and matching of moments (Krilov subspaces) [11], [22], [27], [30-34] etc. which retained desired structure in ROM. Researchers came across ambiguity that in Krilov, Arnoldi, or moment-matching contrivances, ROM got instable in the process. Also, a-prior trimming err bounds found extinct. Nevertheless stability factor there for ROM is maintained and so does the a-prior reduction err bound for SOETM [29-30], [35-37]. Remarkable contribution by Stykel has been found for stable systems [23]. Occasional practical requirements concerning physical systems need accreditation for realization of ROM to intended freq-hiatus exclusively. This further leads to the evolution of finite frequency Model Order Reduction (MOR) methods, specifically Low-Frequency MOR (LFMOR) techniques. [14], [38-40]. In LFMOC, realization of ROM pointed on an expressed freq-interval for stable systems brought in to equate controlability along with observability gramianns pointed on an expressed freqinterval is established. The intellection for LFMOC for standardized systems as presented by [28], which was further modified by [41] for generalized systems. Also, marvellous contribution of the author, on the SOBT arrangement. The LFMOC of discrete and continuous time stable SOSSs using SOETM [38], [42] was produced.

For stable systems, the mentioned techniques works very well but the instability is a major challenge to be faced when CALEs (5) and (6) get mathematically inadequate for controlability and observability gramianns. Miscellaneous MOR approaches [16], [43-51] etc. do not care for unstable systems. Also, MOR of unstable SOSSs is not taken into account at all. Provision of structural retention or physiological position keeping in ROM is not a point of concern, analysis and design process conduction is not permitted and hence no use of ROM left behind. By foreseeing, this research opportunity, edifice-aware MOR procedures for

LFMOR approaches tending to address unstable SOSSs are considered. In the stabilizing step unstable SOSS, Bernoulian stability feeback (BSF) and LFGs are elucidated. The stable SOSS is invoked in CALEs to get gramianns for the equated reduction. The Cholesky factorization of LFGs is obtained, ensuring a coherent solution to CALEs. Procedural gramianns split in pos-velo snippets that pave the way of maintaining the structure in ROM. In addition, the acquired gramianns are equated differently with relent SOETM techniques. Final ROM with SOSS exposition in focused freq-hiatus is achieved. observations for certain cases like [52] and some others are shown and discussed at length. Paper is presented as upcoming section endorses fundamentals of unstable and stable SOSSs succeeded by proposed techniques in the section 4. Discussion of results takes place in the next section and study us concluded.

III. Second-Orderly Structured Systems

For SOSS of (1), the transfer function is presented below:

$$\mathrm{G}(s) = (sCs_2 + Cs_1)(s^2\mu_s\eta_s + \mathrm{K}_s)^{-1}\beta s_2 \quad (7)$$

or summarized as $\mathrm{G}(s) = [\mu_s, \eta_s, \mathbf{K}_s, \beta s_2, Cs_1, Cs_2]$. System (1) of $P(\lambda) = \lambda^2\mu_s + \lambda\eta_s + \mathrm{K}_s$ where as stability depends upon zeros lying in the left or right half planes $P(\lambda) = \lambda^2\mu_s + \lambda\eta_s + \mathrm{K}_s$ have positive real roots. The observability and controlability criteria for system (1) and (3) are explained in [23]. Considering system (1) in an equivalent structural form X-formation (T_l, T_r) with T_l and T_r are taken non-singular as given in (8).

$$\begin{matrix} \overline{\mu_s} = T_l\mu_s T_r & \bar{\eta}_s = T_l\eta_s T_r & \overline{\mathcal{K}}_s = T_l\mathbf{K}_s T_r \\ \bar{B}_2 = T_l B_2 & \bar{C}s_1 = Cs_1 T_r & \bar{C}s_2 = Cs_2 T_r \end{matrix} \quad (8)$$

and first order transformed form of (3):

$$\bar{\gamma} = \bar{T}_l \curlyvee \bar{T}_r \; \bar{\alpha} = \bar{T}_l\alpha\bar{T}_r \; \bar{\beta}_s = \bar{T}_l B \; \bar{C}s = Cs\bar{T}_r$$

where $\bar{T}_l = \begin{bmatrix} T_r^{-1} & 0 \\ 0 & T_l \end{bmatrix}, \bar{T}_r = \begin{bmatrix} T_r & 0 \\ 0 & T_r \end{bmatrix}$.

If we consider system (1) as stable then observability and controlability gramianns for system (3) given by:

$$P_{c.} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (j\hat{\Omega}\curlyvee - \alpha)^{-1}BB^T(-j\hat{\Omega}\curlyvee - \alpha)^{-T}d\hat{\Omega} \quad (9)$$

$$Q_{o.} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (-j\hat{\Omega}\curlyvee - \alpha)^{-T}C_s^T C_s(j\hat{\Omega}\curlyvee - \alpha)^{-1}d\hat{\Omega} \quad (10)$$

The solutions for generalized CALEs (5) and (6) are semidefinite and positive, unique and symmetric. We get positive definite controlability and observability gramianns when (3) is stable using equations (5) and (6).

The gramianns are +ve semdefinitive and hence (3) is unstable, CCALEs of (5) and (6) do not relent +ve definitive controlability and observability gramianns (9) and (10) that hinders the curtailment procedure for unstable SOSSs over unbounded or bounded appliances of frequency hiatus. A reduction procedure proposed in next portion, avert this issue. This contrivance stabilizes the pristine unstable system in the first go and then considers proposed LFMOC contrivance to obtain required ROMs optimized for bounded frequency appliances of unstable SOSSs.

IV. Proposed Methodology

For the CALEs of (5) and (6) to be solvable for unstable SOSSs, Bernouli stabilization solution is applied on pristine system and for emphasizing ROM accomplishment in required frequency hiatus, LFGs for stable SOSS are defined with respective bounded frequency CALEs. An optimized solution for bounded frequency CALEs is given, utilizing computed LFGs in SOETM frameworks, where frequency gap position, velocity, or both are adjusted to derive the ROMs in this section.

A. Stabilization of Unstable SOSS

The CALEs (11) and (12) proposed in [53] to compute the generalized observability and controlability gramianns $Q_{o.}; P_{c.}$ for unstable generalized system are given below:

$$(\alpha - BK_c)P_{c\hat{t}b}\curlyvee^T + \curlyvee P_{cs\hat{t}b}(\alpha - BK_c)^T = -BB^T \quad (11)$$

$$(\alpha - K_oC_s)^T Q_{os\hat{t}b}\curlyvee + \curlyvee^T Q_{os\hat{t}b}(\alpha - K_oC_s) = -C_s^T C_s \quad (12)$$

where matrices $K_c = B^T X_c \curlyvee$ and $K_o = \curlyvee X_o C_s^T$ represents Bernouli stabilizing feedback. Xo and Xc represent stable solutions of the general algebraic Bernouli equations (13) and (14) respectively.

$$\curlyvee^T X_c\alpha + \alpha^T X_c\curlyvee = \alpha^T X_c BB^T X_c\curlyvee \quad (13)$$

$$\alpha X_o\curlyvee^T + \alpha X_o\alpha^T = \curlyvee X_o C_s^T C_s X_o\curlyvee^T: \quad (14)$$

The retention of structure of (3) in [53] has not been catered for, but in our context, we need to ensure the fulfillment of structure retention restraint which is achieved by partitioning of LFGs in next section. Once the above stabilized system is obtained, LFGs are computed for emphasizing ROM accomplishment for LF applications next.

B. Bounded Frequency Gramianns

The bounded frequency observability and controlability gramianns for stabilized SOSS are given in (15) and (16).

$$P_{\hat{\delta}cs\hat{t}b} = \frac{1}{2\pi}\int_{\hat{\delta}} \left(j\hat{\Omega}\curlyvee - \alpha_{cs\hat{t}b}\right)^{-1}BB^T\left(-j\hat{\Omega}\curlyvee - \alpha_{cs\hat{t}b}\right)^{-T}d\hat{\Omega} \quad (15)$$

$$Q_{\hat{\delta}os\hat{t}b} = \frac{1}{2\pi}\int_{\hat{\delta}} (-j\hat{\Omega}\curlyvee - Aos\hat{t}b)^{-T}C_s^T C_s(j\hat{\Omega}\curlyvee - \alpha os\hat{t}b)^{-1}d\hat{\Omega} \quad (16)$$

where $\hat{\delta}$ is the symmetric frequency hiatus $\hat{\delta} = [-\hat{\Omega}_2, -\hat{\Omega}_1] \cup [\hat{\Omega}_1, \hat{\Omega}_2]$ and the LFGs $P_{\hat{\delta}cs\hat{t}b}$ and $Q_{\hat{\delta}os\hat{t}b}$ are real, symmetric and positive definite, $\alpha_{cc\hat{s}t} = \alpha - BK_c, \alpha_{os\hat{t}b} = \alpha - K_oC_s^T$. When $\hat{\Omega}_2 = +\infty$ and $\hat{\Omega}_1 = -\infty$ the gramianns depicted in (15) and (16) are converted similar to (9) and (10) respectively.

Proposition 1: The FLGs $P_{\hat{\delta}cs\hat{t}b}$ and $Q_{\hat{\delta}os\hat{t}b}$ of (15) and (16) are solutions of the bounded frequency CALEs (17) and (18):

$\curlyvee P_{\hat{\delta}cs\hat{t}b}\alpha^T_{cs\hat{t}b} + \alpha_{cs\hat{t}b}P_{\hat{\delta}cs\hat{t}b} \curlyvee^T = -\curlyvee H_c BB^T - BB^T H_c^T \curlyvee^T,$

$$\curlyvee^T Q_{\hat{\delta}os\hat{t}b}\alpha_{os\hat{t}b} + \alpha^T_{os\hat{t}b}Q_{\hat{\delta}os\hat{t}b}\curlyvee = -\curlyvee^T H_o^T C_s^T C_s - C_s^T C_s H_o \curlyvee \quad (17)$$

Respectively. Where $B, C_s, \curlyvee, \alpha_{cs\hat{t}b}$ and $\alpha_{os\hat{t}b}$ hold secondorderly structure structure and

$$H_c = \frac{1}{2\pi}\int_{\hat{\delta}}\left(j\hat{\Omega}\curlyvee - \alpha_{cs\hat{t}b}\right)^{-1}d\hat{\Omega}, H_o = \frac{1}{2\pi}\int_{\hat{\delta}}\left(j\hat{\Omega}\curlyvee - \alpha_{os\hat{t}b}\right)^{-1}d\hat{\Omega}$$

and using Weierstrass canonical form [54] we have:

$$\curlyvee^T H_o^T \alpha^T_{os\hat{t}b} = \frac{1}{2\pi}\int_{\hat{\delta}}\curlyvee^T\left(j\hat{\Omega}\curlyvee - \alpha_{os\hat{t}b}\right)^{-T}\alpha^T_{os\hat{t}b}d\hat{\Omega} = \frac{1}{2\pi}\int_{\hat{\delta}}\alpha^T_{os\hat{t}b}\left(j\hat{\Omega}\curlyvee - \alpha_{os\hat{t}b}\right)^{-T}\curlyvee^T d\hat{\Omega} = \alpha^T_{os\hat{t}bb}H_o^T\curlyvee^T$$

Substituting above equivalence in (21) we get:

$$\curlyvee^T Q_{\hat{\delta}os\hat{t}bb}\alpha_{os\hat{t}b} + \alpha^T_{os\hat{t}b}Q_{\hat{\delta}os\hat{t}b}\curlyvee = \curlyvee^T H_o^T\left(\curlyvee^T Q_o\alpha_{os\hat{t}b} + \alpha^T_{os\hat{t}b}Q_o\curlyvee\right) + \left(\curlyvee^T Q_o\alpha_{os\hat{t}b} + \alpha^T_{os\hat{t}bb}Q_o\curlyvee\right)H_o\curlyvee = -\curlyvee^T H_o^T C_s^T C_s - C^T C_s H_o\curlyvee$$

Similar proof are given for (17)

Partitioning of LFGs (15) and (16) yields:

$$P_{\hat{\delta}cs\hat{s}b} = \begin{bmatrix} P_{pc\hat{\delta}s\hat{\delta}b} & P_{12c\hat{\delta}\hat{\delta}t} \\ P^T T2\hat{c}\hat{s}\hat{t}b & P_{vc\hat{s}\hat{t}b}\end{bmatrix},$$

$$Q_{\hat{\delta}os\hat{t}b} = \begin{bmatrix} Q_{\text{poosstb}} & Q_{12o\hat{\delta}stb} \\ Q^T_{12o\hat{\delta}s\hat{t}b} & Q_{vo\hat{\delta}\hat{t}b}\end{bmatrix}$$

Proof: Consider the conjugate of H_o is exactly same to H_o itself [41]:

$$\tilde{H}_o = \frac{1}{2\pi}\int_{-\hat{\Omega}_2}^{-\hat{\Omega}_1}\left(-j\hat{\Omega}\curlyvee - \alpha_{os\hat{t}b}\right)^{-1}d\hat{\Omega} + \frac{1}{2\pi}\int_{\hat{\Omega}_1}^{\hat{\Omega}_2}\left(-j\hat{\Omega}\curlyvee - \alpha_{os\hat{t}b}\right)^{-1}d\hat{\Omega}$$

$$= \frac{1}{2\pi}\int_{\hat{\Omega}_1}^{\hat{\Omega}_2}\left(j\hat{\Omega}\curlyvee - \alpha_{os\hat{t}b}\right)^{-1}d\hat{\Omega} + \frac{1}{2\pi}\int_{-\hat{\Omega}_2}^{-\hat{\Omega}_1}\left(-j\hat{\Omega}\curlyvee - \alpha_{os\hat{t}b}\right)^{-1}d\hat{\Omega} = H_o$$

Next, the hiatus gramiann (10) and LFG (16) relation is obtained from (12) as:

$$C_s^T C_s = -\curlyvee^T Q_o\alpha_{os\hat{t}b} - \alpha^T_{os\hat{t}b}Q_o\curlyvee = \curlyvee^T Q_o\left(-j\hat{\Omega}\curlyvee - \alpha_{os\hat{t}b}\right) + \left(j\hat{\Omega}\curlyvee^T - \alpha^T_{os\hat{t}b}\right)Q_o\curlyvee$$

Pre and post multiplication of above relation with $\left(j\hat{\Omega}\curlyvee^T - \alpha_{os\hat{t}b}\right)^{-1}$ and $\left(-j\hat{\Omega}\curlyvee^T - \alpha^T_{os\hat{t}b}\right)^{-1}$ respectively and integrating over $\hat{\delta}$ yield:

$$Q_{\hat{\delta}os\hat{t}b} = H_o^T\curlyvee^T Q_o + Q_o\curlyvee H_o \quad (19)$$

Similarly, using (15), (9) and (11) we get:

$$P_{\hat{\delta}c\hat{t}b} = H_c\curlyvee P_c + P_c\curlyvee^T H_c^T \quad (20)$$

From (19) we write:

$$\curlyvee^T Q_{\hat{\delta}os\hat{t}b}\alpha_{os\hat{t}b} + \alpha^T_{os\hat{t}b}Q_{\hat{\delta}os\hat{t}b}\curlyvee = +\alpha^T_{os\hat{t}b}H_o^T\curlyvee^T Q_o\curlyvee + \alpha^T_{os\hat{t}b}Q_o\curlyvee H_o\curlyvee$$

where $Q_{vo\hat{\delta}\hat{t}b}, Q_{po\hat{\delta}s\hat{t}b}, P_{vc\hat{\delta}\hat{t}b}$ and $P_{pc\hat{\delta}\hat{\delta}\hat{t}b}$ are bounded frequency velocity and position observability and controlability gramianns respectively. As defined in 1, for the bounded frequency hiatus, the product of the factors is used to get the HSVs for second-orderly structured equated reduction.

C. Performance Measure

It is important to define the performance metric used in the process of reduction before discussing further. In the model reduction, the difference between pristine and following curtailed model is the error performance measure. Which is equated by the norms as in [16] or by averaged error as in [38]. Therefore, the absolute average error is given as:

$$e_k = \left|\sum_{i=1}^{N}\left(\frac{y_{ki} - y_{kri}}{N}\right)\right| \quad (22)$$

while, $k = 1 \ldots p, i = 1 \ldots N$ $N =$ no. of output samples in the considered duration of time by using respective contrivances as given in algorithms.

Definition 1: Consider the stabilized SOSS.

1. The sqrt. of eigenvalues of product $P_{pc\hat{\delta}s\hat{t}b}Q_{po\hat{\delta}s\hat{t}b}$ yields position Hankel Singular Values (HSVs).
2. The sqrt. of eigenvalues of product $P_{vc\hat{\delta}s\hat{t}b}Q_{vo\hat{\delta}s\hat{t}b}$ yields the velocity HSVs.
3. The sqrt. of eigenvalues of the product $P_{pc\hat{\delta}s\hat{t}b}Q_{vo\hat{\delta}s\hat{t}b}$ yield position-velocity HSVs.
4. The sqrt. of eigenvalues of product $P_{vc\hat{\delta}s\hat{t}b}Q_{po\hat{\delta}s\hat{t}b}$ yields the velocity-position HSVs.

Remark 1: The partitioning of gramianns into portions of velocity and position ensures the retention of SOSS structure in ROM. Hence interpretation of SOSS in ROM is retained and tools designed for SOSSs can be applied to the ROMs.

D. Bounded Frequency Equated Transformation

For HSVs, different methods are applied to system (1) to compute the position and velocity.

Definition 2: For the stable system and $\hat{\delta}$:

1. The position is used for equating the system when $P_{pc\hat{\delta}\hat{s}b} = Q_{po\hat{\delta}s\hat{t}b} = \text{diagnn}\left(\hat{\zeta}_1^{p\hat{s}\hat{t}b}, \ldots, \hat{\zeta}_n^{p\hat{\delta}s\hat{t}b}\right)$.
2. The system is velocity equated when $P_{vc\hat{\delta}s\hat{t}bb} = Q_{vo\hat{\delta}s\hat{t}b} = \text{diagnn}\left(\hat{\zeta}_1^{v\delta s\hat{\hat{t}}bb}, \ldots, \hat{\zeta}_n^{v\delta s\hat{t}bb}\right.$.
3. The system is position-velocity equated when $P_{pc\hat{\delta}\hat{s}tb} = Q_{vo\hat{\delta}s\hat{t}b} = \text{diagnn}\left(\hat{\zeta}_1^{pv\hat{\delta}s\hat{t}b}, \ldots, \hat{\zeta}_n^{pv\hat{\delta}\hat{t}b}\right)$.
4. The system is velocity-position equated when $P_{vc\hat{\delta}\hat{s}tb} = Q_{po\hat{\delta}s\hat{t}b} =$

diagnn $\left(\hat{\zeta}_1^{vp\hat{\delta}st b}, \ldots, \hat{\zeta}_n^{vp\hat{\delta}\hat{t}b}\right)$.

where $\hat{\zeta}$ are HSVs for bounded hiatus in the decreasing order. The Cholesky factors of LFGs are calculated to get equated transformation (23):

$$P_{pc\hat{\delta}\hat{t}b} = \mathbb{R}_{p\hat{\delta}}\mathbb{R}_{p\hat{\delta}}^T \quad P_{vc\hat{\delta}s\hat{t}b} = \mathbb{R}_{v\hat{\delta}}\mathbb{R}_{v\hat{\delta}}^T$$
$$Q_{po\hat{\delta}s\hat{t}b} = L_{p\hat{\delta}}L_{p\hat{\delta}}^T \quad Q_{vo\hat{\delta}s\hat{t}b} = L_{v\hat{\delta}}L_{v\hat{\delta}}^T \tag{23}$$

while, the Cholesky factors $\left(\mathbb{R}_{v\hat{\delta}}, \mathbb{R}_{p\hat{\delta}}, L_{p\hat{\delta}}, L_{v\hat{\delta}} \in \mathbb{R}^{n\times n}\right)$ are utilized to compute the HSVs using singular values (SVs) as detailed below:

$$\left(\hat{\zeta}_i^{p\hat{\delta}stb}\right)^2 = \lambda_i\left(P_{p\hat{\delta}\hat{\delta}\hat{b}}Q_{po\hat{\delta}\hat{t}bb}\right) = \lambda_i\left(\mathbb{R}_{p\hat{\delta}}\mathbb{R}_{p\hat{\delta}}^T L_{p\hat{\delta}}L_{p\hat{\delta}}^T\right)$$
$$= \lambda_i\left(L_{p\hat{\delta}}^T\mathbb{R}_{p\hat{\delta}}\mathbb{R}_{p\hat{\delta}}^T L_{p\hat{\delta}}\right) = \sigma_i^2\left(\mathbb{R}_{p\hat{\delta}}L_{p\hat{\delta}}\right)$$

while $\sigma_{i\hat{\delta}}$'s are the SVs.

Likewise, $\hat{\zeta}_i^{v\hat{\delta}s\hat{t}b} = \sigma_i\left(\mathbb{R}_{v\hat{\delta}}^T\mu_s^T L_{v\hat{\delta}}\right)$ and $\hat{\zeta}_i^{vp\hat{\delta}s\hat{t}b} = \sigma_i\left(\mathbb{R}_{v\hat{\delta}}^T L_{p\hat{\delta}}\right)$. SVD of these products yield (24).

$$\mathbb{R}_{p\hat{\delta}}^T L_{p\hat{\delta}} = U_{p\hat{\delta}}\Sigma_{p\hat{\delta}}V_{p\hat{\delta}}^T \quad \mathbb{R}_{v\hat{\delta}}^T\mu_s^T L_{v\hat{\delta}} = U_{v\hat{\delta}}\Sigma_{v\hat{\delta}}V_{v\hat{\delta}}^T$$
$$\mathbb{R}_{p\hat{\delta}}^T\mu_s^T L_{v\hat{\delta}} = U_{pv\hat{\delta}}\Sigma_{pv\hat{\delta}}V_{pv\hat{\delta}}^T \quad \mathbb{R}_{v\hat{\delta}}^T L_{p\hat{\delta}} = U_{vp\hat{\delta}}\Sigma_{vp\hat{\delta}}V_{vp\hat{\delta}}^T \tag{24}$$

while $U_{vp\hat{\delta}}, V_{v\hat{\delta}}, U_{p\hat{\delta}}, V_{p\hat{\delta}}, U_{v\hat{\delta}}, V_{vp\hat{\delta}}, U_{pv\hat{\delta}}, V_{pv\hat{\delta}}$, are the orthogonal matrices and

$$\Sigma_{p\hat{\delta}} = \text{diagnn}\left(\hat{\zeta}_1^{p\hat{\delta}s\hat{\delta}b}, \ldots, \hat{\zeta}_n^{p\hat{\delta}stbb}\right)$$
$$\Sigma_{v\hat{\delta}} = \text{diagnn}\left(\hat{\zeta}_1^{v\hat{\delta}s\hat{t}b}, \ldots, \hat{\zeta}_n^{v\hat{\delta}stb}\right)$$
$$\Sigma_{pv\hat{\delta}} = \text{diagnn}\left(\hat{\zeta}_1^{pv\hat{\delta}stb}, \ldots, \hat{\zeta}_n^{pv\hat{\delta}s\hat{t}b}\right)$$
$$\Sigma_{vp\hat{\delta}} = \text{diagnn}\left(\hat{\zeta}_1^{vp\hat{\delta}stb}, \ldots, \hat{\zeta}_n^{vp\hat{\delta}\hat{s}b}\right)$$

are non-singular matrices. For the methods elaborated by definition 2, relation (24) is utilized to get the LF second orderly equated transformation $\left(P_{l\hat{\delta}}, P_{r\hat{\delta}}\right)$.

E. Bounded Frequency Second-orderly Structured Equated Reduction

The role of system velocity and position in dynamics is depicted by the magnitude of HSVs. To get ROM, the smaller states are trimmed at the cost of small error. Based on our earlier discussion, we are highlighting two different algorithms for bounded frequency position and velocity equated trimming. $P_{l\hat{\delta}}$ is the transformation matrices, given in algorithm 1, that are selected as given below:

$$P_{pc\hat{\delta}\hat{t}b} = Q_{po\hat{\delta}s\hat{t}b} = Q_{vo\hat{\delta}s\hat{t}b}$$
$$= \text{diag}\left(\hat{\zeta}_1^{p\hat{\delta}s\hat{t}b}, \ldots, \hat{\zeta}_r^{p\hat{s}\hat{t}b}\right)$$

while in algorithm 2 equating of velocity observability gramiann along with position controlability gramiann.

Algorithm 1: The position equating based Bounded Frequency SOETM (LFSOETMp)

In: For the any large scale stabilized SOSS G = [$\mu_s, \eta_s, K, B_2, Cs_1, Cs_2$] and frequency hiatus $\hat{\delta}$:

Out: The bounded frequency ROM

$\mathrm{G}_r = [\mu_{sr}, \eta_{sr}, \mathrm{K}_{sr}, \beta s_{2r}, Cs_{1r}, Cs_{2r}]$

1. Compute the LFGs $P_{pc\hat{\delta}\hat{s}b}, Q_{po\hat{\delta}\hat{s}b}, P_{vc\hat{\delta}s\hat{t}b}$ and $Q_{vo\hat{\delta}s\hat{t}b}$ using (17) and (18).
2. Compute the Cholesky factors $\mathbb{R}_{v\hat{\delta}}, \mathbb{R}_{p\hat{\delta}}, L_{p\hat{\delta}}$ and $L_{v\hat{\delta}}$ of LFGs using (24).
3. Compute SVD for products:

$$\mathbb{R}_{p\hat{\delta}}^T L_{p\hat{\delta}} = \begin{bmatrix}U_{p1\hat{\delta}} & U_{p2\hat{\delta}}\end{bmatrix}\begin{bmatrix}\Sigma_{p1\hat{\delta}} & 0\\ 0 & \Sigma_{p2\hat{\delta}}\end{bmatrix}\begin{bmatrix}V_{p1\hat{\delta}} & V_{p2\hat{\delta}}\end{bmatrix}^T$$
$$\mathbb{R}_{v\hat{\delta}}^T\mu_s^T L_{v\hat{\delta}} = \begin{bmatrix}U_{v1\hat{\delta}} & U_{v2\hat{\delta}}\end{bmatrix}\begin{bmatrix}\Sigma_{v1\hat{\delta}} & 0\\ 0 & \Sigma_{v2\hat{\delta}}\end{bmatrix}\begin{bmatrix}V_{v1\hat{\delta}} & V_{v2\hat{\delta}}\end{bmatrix}^T$$

while $[U_{p1\hat{\delta}} \quad U_{p2\hat{\delta}}], [V_{p1\hat{\delta}} \quad V_{p2\hat{\delta}}], [U_{v1\hat{\delta}} \quad U_{v2\hat{\delta}}]$ and $[V_{v1\hat{\delta}} \quad V_{v2\hat{\delta}}]$ are orthogonal matrices and

$$\Sigma_{p1\hat{\delta}} = \text{diagnn}\left(\hat{\zeta}_1^{p\hat{\delta}s\hat{t}b}, \ldots, \hat{\zeta}_r^{p\hat{\delta}s\hat{t}b}\right)$$
$$\Sigma_{p2\hat{\delta}} = \text{diagnn}\left(\hat{\zeta}_r^{p\hat{s}\hat{t}b}, \ldots, \hat{\zeta}_n^{p\hat{\delta}stb}\right)$$
$$\Sigma_{v1\hat{\delta}}^p = \text{diagnn}\left(\hat{\zeta}_1^{v\hat{\delta}\hat{t}b}, \ldots, \hat{\zeta}_r^{v\hat{\delta}s\hat{t}b}\right)$$
$$\Sigma_{v2\hat{\delta}}^v = \text{diagnn}\left(\hat{\zeta}_{r_+}^{v\hat{\delta}satb}, \ldots, \hat{\zeta}_n^{v\hat{\delta}\hat{t}b}\right)$$

4. Calculation of the ROM

$$\mu_s P_{r\hat{\delta}} \qquad D_r = P_{l\hat{\delta}}^T D P_{r\hat{\delta}} \qquad \mathrm{K}_{sr} = P_{l\hat{\delta}}^T\,\mathrm{K}_s P_{r\hat{\delta}}$$
$$B_{2r} = P_{l\hat{\delta}}^T B_2 \quad Cs_{1r} = Cs_1 P_{r\hat{\delta}} \quad Cs_{2r} = Cs_2 P_{r\hat{\delta}}$$

while $P_{r\hat{\delta}} = \mathbb{R}_{p\hat{\delta}}U_{p1\hat{\delta}}\Sigma_{p1\hat{\delta}}^{-1/2}$ and $P_{l\hat{\delta}} = L_{v\hat{\delta}}V_{v1\hat{\delta}}\Sigma_{p1\hat{\delta}}^{-1/2}$

Algorithm 2: The Position and Velocity Equated Bounded Frequency SOETM (LFSOETMpv)

In: For stable large SOSS and frequency hiatus $\hat{\delta}$:

Out: ROM (frequency bounded)

$\mathrm{G}_r = [\mu_{sr}, \eta_{sr}, \mathrm{K}_{sr}, B_{2r}, Cs_{1r}, Cs_{2r}]$

1. Compute the LFGs $P_{pc\hat{\delta}s\hat{t}b}$ and $Q_{vo\hat{\delta}s\hat{t}b}$ applying (17) and (18).
2. Compute the Cholesky factors $L_{v\hat{\delta}}$ and $\mathbb{R}_{p\hat{\delta}}$ of LFGs $P_{pc\hat{\delta}s\hat{t}b}$ and $Q_{vo\hat{\delta}s\hat{t}b}$ as explained in (23).
3. Compute the SVD for products:

$$\mathbb{R}_{p\hat{\delta}}^T\mu_s^T L_{v\hat{\delta}} = \begin{bmatrix}U_{pv1\hat{\delta}} & U_{pv2\hat{\delta}}\end{bmatrix}\begin{bmatrix}\Sigma_{pv1\hat{\delta}} & 0\\ 0 & \Sigma_{pv2\hat{\delta}}\end{bmatrix}\begin{bmatrix}V_{pv1\hat{\delta}} & V_{pv2\hat{\delta}}\end{bmatrix}^T$$

while $[V_{pv1\hat{\delta}} \quad V_{pv2\hat{\delta}}]$ and $[U_{pv1\hat{\delta}} \quad U_{pv2\hat{\delta}}]$ are orthogonal and

$$\Sigma_{pv1\hat{\delta}} = \text{diagnn}\left(\hat{\zeta}_1^{pv\hat{\delta}s\hat{t}b}, \ldots, \hat{\zeta}_r^{pv\hat{\delta}s\hat{t}b}\right)$$
$$\Sigma_{pv2\hat{\delta}} = \text{diagnn}\left(\hat{\zeta}_{r_+}^{pv\hat{\delta}s\hat{t}b}, \ldots, \hat{\zeta}_n^{pv\hat{\delta}st\hat{b}b}\right)$$

4. Calculate the ROM

$$\mu_{sr} = I_n \qquad \eta_{sr} = P_{l\hat{\delta}}^T\eta_s P_{r\hat{\delta}} \quad \mathbf{K}_{sr} = P_{l\hat{\delta}}^T\mathbf{K}_s P_{r\hat{\delta}}$$
$$\beta s_{2r} = P_{l\hat{\delta}}^T B_2 \quad Cs_{1r} = Cs_1 P_{r\hat{\delta}} \quad Cs_{2r} = Cs_2 P_{r\hat{\delta}}$$

where $P_{r\hat{\delta}} = \mathbb{R}_{p\hat{\delta}}U_{pv1\hat{\delta}}\Sigma_{pv1\hat{\delta}}^{-1/2}$ and $P_{l\hat{\delta}} = L_{v\hat{\delta}}V_{pv1\hat{\delta}}\Sigma_{pv1\hat{\delta}}^{-1/2}$

F. Cholesky Factors and Cales Solution

For LFGs, the solution that arise from (17) and (18) is computationally extensive, and to obtain Cholesky factors gets difficult for indefinite RHS of (17) and (18) [41]. Hence, we used quadrature rule to assign weights for efficient approximation of (17) and (18), presented as: ξ_i to the nodes $\hat{\Omega}_i$ [55]. It gives solution at various points within the bounded frequency range. As highlighted in [41], the Gauss quadrature rule outperforms the Boole and trapezoidal rules, therefore, we employed the Gauss quadrature rule for computing the solution. Below is the approximated LFG for the equation (17).

$$P_{\hat{\delta}cs\hat{t}b} \approx \frac{1}{2\pi}\sum_{i=1}^{K_s} \xi_i \left\{\left(j\hat{\Omega}_i \curlyvee -A\right)^{-1} BB^T\left(-j\hat{\Omega}_i \curlyvee -A\right)^{-T} + \right. \quad (25)$$
$$\left.\left(-j\hat{\Omega}_i \curlyvee -A\right)^{-1} BB^T\left(j\hat{\Omega}_i \curlyvee -A\right)^{-T}\right\}$$

Lets consider ξ_i as positive, (25) can be written as (26):

$$P_{\hat{\delta}cs\hat{t}b} \approx \frac{1}{2}\left[B_{1\hat{\delta}}, B_{1\hat{\delta}}^-, \ldots, B_{K_s\hat{\delta}}, B_{K_s\hat{\delta}}^-\right]\left[B_{1\hat{\delta}}, B_{1\hat{\delta}}^-, \ldots, B_{K_s\hat{\delta}}, B_{K_s\hat{\delta}}^-\right]^{[0} \quad (26)$$

along $B_{i\hat{\delta}} = (\xi_i/\pi)^{1/2}\left(j\hat{\Omega}_i \curlyvee -\alpha\right)^{-1} B$. The low rank Cholesky factor is given by relation (26). Furthermore, we can write (26) in product form as: $\left[B_{i\hat{\delta}}, B_{i\hat{\delta}}^-\right]\left[B_{i\hat{\delta}}, B_{i\hat{\delta}}^-\right]^* = 2\left[\mathrm{Re}\left(B_{i\hat{\delta}}\right), \mathrm{Img}\left(B_{i\hat{\delta}}\right)\right] \times \left[\mathrm{Real}\left(B_{i\hat{\delta}}\right), \mathrm{Img}\left(B_{i\hat{\delta}}\right)\right]^T$ this give Cholesky factors $P_{\hat{\delta}cs\hat{t}b} \approx \mathbb{R}_{p\hat{\delta}}\mathbb{R}_{p\hat{\delta}}^T$ while

$$\mathbb{R}_{p\hat{\delta}} = \left[\mathrm{Real}(B_{1\hat{\delta}}), \ldots, \mathrm{Real}(B_{K_s\hat{\delta}}), \mathrm{Img}(B_{1\hat{\delta}}), \ldots, \mathrm{Img}(B_{K_s\hat{\delta}})\right] \quad (27)$$

Similarly, observability gramiann for bounded frequency hiatus is approximated by $Q_{\hat{\delta}ostb} \approx L_{p\hat{\delta}}L_{p\hat{\delta}}^T$ where

$$L_{p\hat{\delta}} = \left[\mathrm{Real}\left(Cs_{1\hat{\delta}}\right), \ldots, \mathrm{Real}\left(Cs_{K_s\hat{\delta}}\right)\right. \quad (28)$$
$$\left., \mathrm{Img}\left(Cs_{1\hat{\delta}}\right), \ldots, \mathrm{Img}\left(Cs_{K_s\hat{\delta}}\right)\right] \quad (29)$$

with $Cs_{i\hat{\delta}} = \sqrt{\xi_i/\pi}\left(-j\hat{\Omega}_i \curlyvee -\alpha\right)^{-1}\Big) Cs^T$.

V. RESULTS AND DISCUSSION

The MOR contrivances that are obtained are solicited to sundry unstable SOSSs such as electromagnetic-system and few self generated simulation examples are given in appendix. Furthermore, their results are also depicted, note that, abbreviation SOET is portrayed as SOBT in all diagrams.

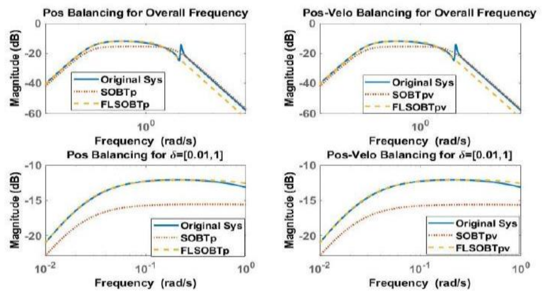

Figure 1: Frequency response for complete range (hiatus $[0.01, 1]\mathrm{rad/s}$) by using the position and the position-velocity for Electromagnetic System

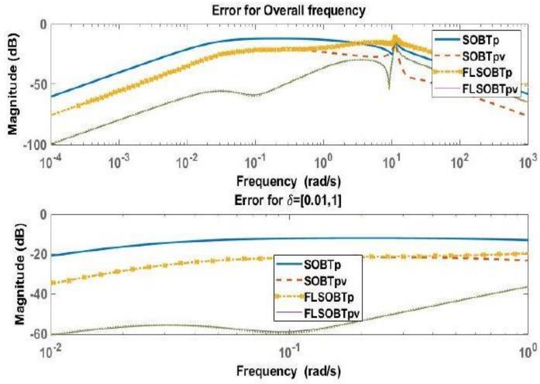

Figure 2: Errors for complete frequency range (hiatus $[0.01, 1]\mathrm{rad/s}$) for Electromagnetic System

A. Electromagnetic System

As depicted in Figure 1. [52], a mechanism for harvesting the electromagnetic energy with interrelated network is utilized between the transmission lines. In the unit length of the loosely coupled single input single output (SISO) interconnect-network at $\xi_0 = 25°\mathrm{C}$, the parameters are

$$R = \frac{\begin{bmatrix} 106 & 0.0 & 0 \\ 0.0 & 1.8 & 0 \\ 0 & 0 & 1.6 \end{bmatrix}\hat{\Omega}}{cm},$$
$$L = \begin{bmatrix} 0.3 & 5.1 & 6.3 \\ 4.1 & 7.3 & 5.1 \\ 9.3 & 5.1 & 1.3 \end{bmatrix} nH/cm,$$
$$C = \frac{\begin{bmatrix} 5.4 & -1.5 & -0.03 \\ -0.4 & 2.4 & -0.3 \\ -0.13 & -4.5 & 2.6 \end{bmatrix} pF}{cm} \quad (29)$$

The model in (29) is an unstable system (SOSS) for 3 levels of RLC-network. The model is reduced to 12 diagonal levels and requested the proposed reduction mechanisms. Using algos 1 and 2, the 12th order is reduced to $r = 1$ system. The bode-riposte for pristine system and for ROM over infinite range and in the hiatus $\hat{\delta} = [0.1,2]\mathrm{rad/s}$ utilizing second-orderly structured equated reduction position (SOETp or SOBTp as in fig.) and second-orderly structured equated reduction positionvelocity (SOETpv) contrivances are depicted in Figure 3. While Figure 2 depicts errors for infinite range of frequency and hiatus $\hat{\delta} = [0.1,2]\mathrm{rad/s}$. Note: the curtailed model is separating the pristine SOSS very closely.

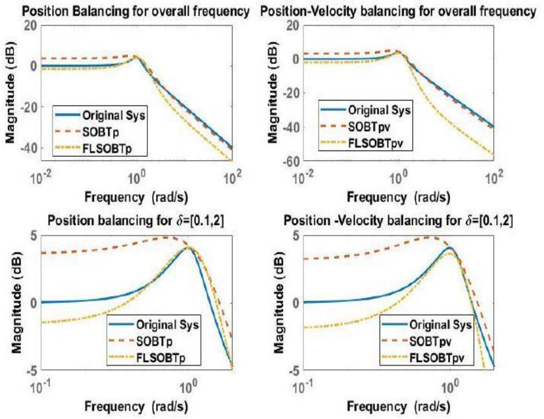

Figure 3: Frequency response for complete range (hiatus $[0.1,2]\mathrm{rad/s}$) utilizing position and the position-velocity equating for example 1

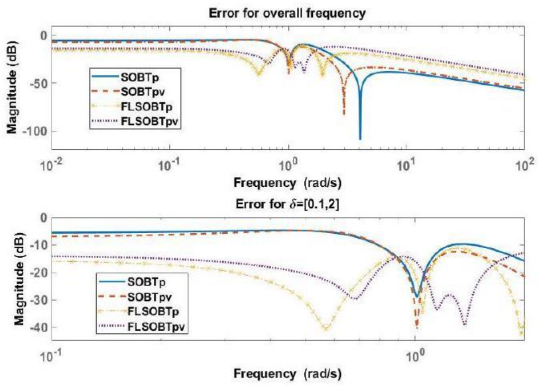

Figure 4: Errors for complete range of frequency (hiatus [0.1,2] rad/s) for example 1

B. Simulation Cases

The proposed MOR mechanisms are evaluated on various unstable SOSSs listed in the Appendix and sequels are collocated with infinite gramiann contrivances for correction the veracious development. Its, also overwhelms the proposed contrivances on the current contrivances to endorse the error less development and dominance.

Single Input Single Output Systems:

Example 1: The suggested algos 1 and 2 are solicited to curtail the 9^{th} order system of example 1 till r of 2 in hiatus $\hat{\delta} = [0.1,2]\mathrm{rad/s}$. In Figure 4, the frequency response for pristine unstable system is given. Furthermore, it also shows ROM using SOETp, LFSOETp, SOETpv and LFSOETpv for infinite range of frequency and for hiatus $\hat{\delta} = [0.1,2]\mathrm{rad/s}$. While the figure 5 depicts errors in infinite range of frequency and errors for hiatus $\hat{\delta} = [0.1,2]\mathrm{rad/s}$. Noteworthy: LFSOETMp and LFSOETMpv algorithms behaves closer to sublimity in bounded hiatus when collocated to infinite hiatus contrivances in the selected hiatus $\hat{\delta} = [0.1,2]\mathrm{rad/s}$.

Example 2: Algos 1 and 2 are applied to curtail the 11^{th} order system of second example till rof3 in hiatus $\hat{\delta} = [0.01,5]\mathrm{rad/s}$. In bode-riposte for pristine system having ROM with LFSOETp, SOETp, LFSOETpv and SOETpv for infinite frequency range and for hiatus $\hat{\delta} = [0.01,5]\mathrm{rad/s}$

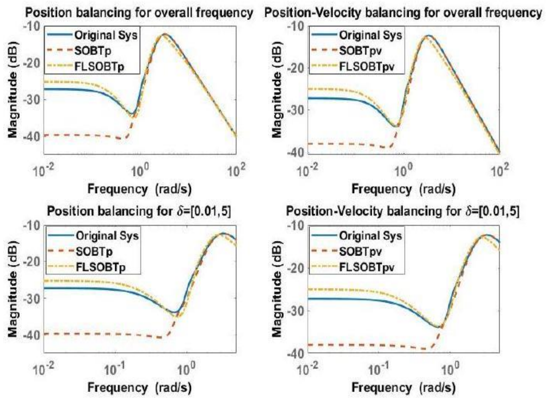

Figure 5: Frequency response for complete range (hiatus $[0.01,5]\mathrm{rad/s}$) utilizing position and the position-velocity equating for example 2

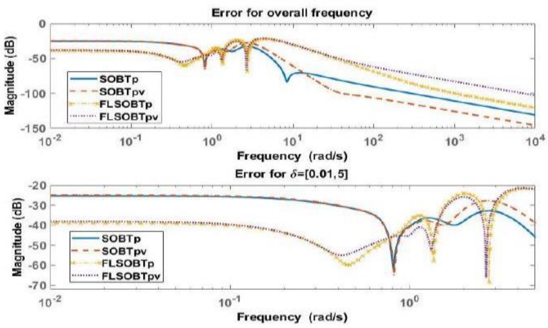

Figure 6: Error for complete frequency range (hiatus [0.01,5]rad/s)

For example 2 is elaborated in Figure 6. The errors for infinite span of frequency and for $\hat{\delta} = [0.01,5]\mathrm{rad/s}$ is given in the Figure 7. For this system, again, it is observed that LFSOETp and LFSOETpv algorithms are very well in performance in given bounded frequency hiatus as collocated to infinite hiatus contrivance.

Multi Input Multi Output System:

Example 3: In the end, represented contrivances are evaluated on multi inputs multi outputs system (MIMO). The system having 9^{th} order in example 3, (having 2 I/Os) is simplified to $r = 5$ in the $\hat{\delta} = [0.01,6.9]\mathrm{rad/s}$ and riposte for pristine unstable 2nd order structured system and the ROMs using SOETMp, LFSOETMp, SOETMpv and LFSOETMpv contrivances is delineated in the Figure 8 for infinite range of frequency. In Figure 9, the riposte for SOETMp, LFSOETMp, SOETMpv and LFSOETMpv contrivances in hiatus $\hat{\delta} = [0.01,6.9]\mathrm{rad/s}$ is portrayed. It is again noticed that the proposed contrivance gives results that are very similar to the pristine system in the desired frequency hiatus. In Figure 7, the errors are plotted for infinite range of frequency and for $\hat{\delta} = [0.01, 6.9]\mathrm{rad/s}$. Suggested contrivances produce least curtailment error in given frequency hiatus as depicted in Figure 10. The results that are given above affirms the proper

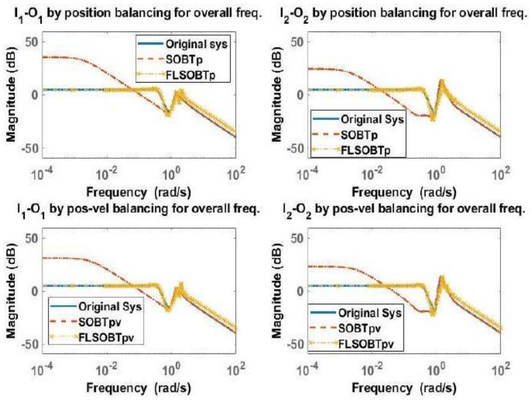

Figure 7: Frequency response of position and position velocity for infinite frequency range for example 3

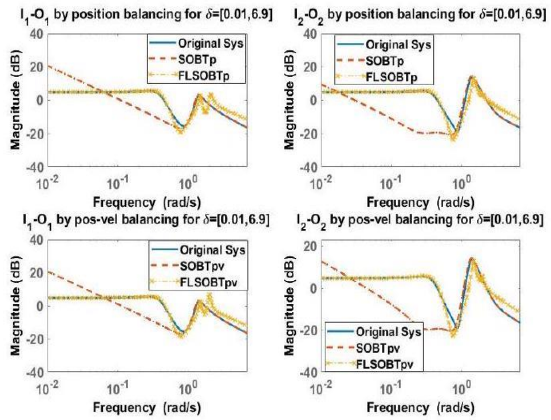

Figure 8: Frequency response of position and position velocity for hiatus $[0.01, 6.9]\mathrm{rad/s}$ for example 3

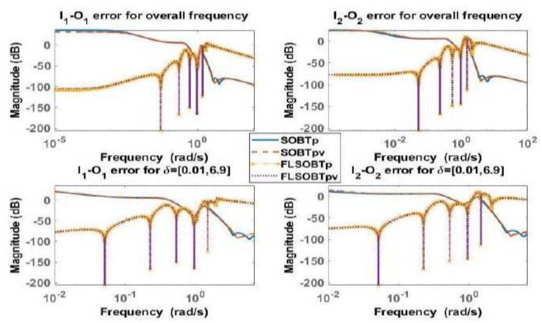

Figure 9: Errors for entire frequency range (hiatus [0.01,6.9]rad/s) for example 3 development of proposed contrivances in bounded range of frequency for MOR applications of unstable SOSSs.

VI. CONCLUSION

Contrivances for the structure that retains MOR of unstable SOSSs with bounded frequency applications is given in this manuscript. The LFGs and respective bounded frequency CALEs are then obtained. We stabilized the unstable SOSS using Bernouli feedback to solve bounded frequency CALEs. It gives us controlability and observability gramianns.

Gramianns are then divided in to position and velocity to attain the system retention. Finally, different coalescences are utilized for equating the gramians to acquire HSVs for curtailment that relent ROMs. The tests are performed on multiple unstable SOSSs for proposed contrivances. Results shows the riposte optimisation in bounded frequency hiatus as well as retention of second-orderly structure in ROM. The presented contrivances are useful MOR Applications with bounded frequency for unstable SOSSs.

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APPENDIX

Example 1: Unstable continuous time 9^{th} order system SISO SOSS

$$K=\begin{bmatrix} 1 & .4 & .2 & 1 & 6 & .7 & .7 & 1.3 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix};$$

$$\hat{\delta}=\begin{bmatrix} .5 & .3 & 0 & 0 & 0 & 1.71 & 0.26 & 0.09 & & 0.38 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 \end{bmatrix}$$

$$\beta_2=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$C_2=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, M=I_9$$

Example 2: An 11th order continuous time unstable continuous SISO SOSS

$$K=\begin{bmatrix} 0 & -4.5 & -23 & -3 & 6.5 & -.6 & -3 & -9 & -.5 & -8 & -2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

$$\hat{\delta}=\begin{bmatrix} 0 & 0 & 0 & .7 & 0 & 0 & 0 & 0 & 0 & 5.1 & 9.3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\beta_2=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$C_1=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$$C_2=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, M=I_{11}$$

Example 3: The 9^{th} order continuous time unstable MIMO SOSS

$$K=\begin{bmatrix} .8 & 0 & .6 & -0.1 & 1 & -0.3 & -0.2 & 0 & 0.32 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix};$$

$$\hat{\delta}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0.13 & 5.7 & 1.1 & 0.03 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

$$\beta_2=\begin{bmatrix} -1 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^T$$

$$C_1=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_2=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, M=I_9$$