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Model Reduction of Unstable Discrete Time Generalized Second Orderly-Form Systems with Structure Preservation

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Abstract- Mathematical models are to be emblematized in equivalently curtailed models. Sundry nonexistent techniques for model reduction of unstable discrete time second-orderly structured systems (SOSs) are suggested. In this paper we consider two structure-retaining model curtailment of discrete time unstable second-orderly systems utilizing equated truncation contrivances. First contrivance fragment the SOSS in stable and unstable parts, while second contrivance utilize Bernouli feedback stabilization stratagem and hence provide structure retention. Assorted collections of singular values are incorporated for such systems, which draws forth different scenarios of balancing and disparate second-orderly equated curtailment stratagems. Characteristics of these techniques are collated and demonstrated through numerical examples. As suggested contrivance maintains second-orderly structure as well as carries stabilized system dynamics therefore, this contrivance truly compares pristine system behaviour. The juxtaposition of suggested contrivances is bestowed for various systems to endorse the veracious development and dominance of suggested contrivances over prevailing contrivances.

Keywords- Second-orderly structured systems, unstable systems, model order reduction, infinite gramianns, Henkel singular values.

I. INTRODUCTION

The mathematical models of physical systems are derived for analysis and design purposes of systems under consideration. The reduced order models (ROM) of these mathematical models are highly demanded because of complexity of these models. ROM is a surrogate model of the pristine model and second-orderly structured systems hold pair of 1st and 2nd state-derivatives in system-dynamics. These systems are found in multifarious fields related to engineering and technology like biological systems, electro-mechanical technology, image processing, community interaction, smart grid systems, huge-buildings, [1-6] etc. The linear time invariant SOSS in discrete-form is emblematized in (1).

$$\mu_s x[k+2] + \eta_s x[k+1] + \underset{s}{\mathrm{K}_s} x[k] = B_{s2} u[k]$$

$$C_{s2} x[k+1] + C_{s1} x[k] = y_s[k]$$

where $\eta_s \in \mathbb{R}_{\sim}^{\;n\times n}, \mu_s \in \mathbb{R}_{\sim}^{\;n\times n}, \beta s_2 \in \mathbb{R}_{\sim}^{\;n\times m}$, $\mathrm{K}_s \in \mathbb{R}_{\sim}^{\;n\times n} Cs_2$ and $Cs_1 \in \mathbb{R}_{\sim}^{p\times n}, x(t) \in \mathbb{R}_{\sim}^{n}, u_s(t) \in \mathbb{R}_{\sim}^{\;m}$ and $y(t) \in \mathbb{R}_{\sim}^{p}, n$ is system order, m is No. of input(s), p is No. of output(s) of the system.

System (1) in generalized form can be written as:

$$\Upsilon\, q[k+1] = \alpha q[k] + B_s u[k]$$

$$y[k] = C_s q[k]$$

with $q[k] = [x[k]^T \quad x[k+1]^T]^T$,

$$\Upsilon = \begin{bmatrix} I_s & 0 \\ 0 & \mu_s \end{bmatrix}, \alpha = \begin{bmatrix} 0 & I_s \\ -\mathrm{K}_s & -\eta_s \end{bmatrix}$$

$$B_s = \begin{bmatrix} 0 \\ B_{s2} \end{bmatrix}, C = [C_{s1} \quad C_{s2}]$$

The goal of model order reduction (MOR) contrivance is to produce a ROM given as:

$$\mu_{sr} x_r[k+2] + \eta_{sr} x_r[k+1] + \mathrm{K}_{sr} x_r[k] = B_{s2r} u[k]$$

$$C_{s2r} x_r[k+1] + C_{s1r} x_r[k] = y_{sr}[k]$$

$\mu_{sr} \in \mathbb{R}_{\sim}^{r\times r}, \eta_{sr} \in \mathbb{R}_{\sim}^{r\times r}, \mathbf{K}_{sr} \in \mathbb{R}_{\sim}^{r\times r}, \beta s_{2r} \in \mathbb{R}_{\sim}^{r\times m}, Cs_{1r}$ and $Cs_{2r} \in \mathbb{R}_{\sim}^{\;p\times r}$,
$x_r(t) \in \mathbb{R}_{\sim}^{r}, u_s(t) \in \mathbb{R}_{\sim}^{m}$ and $y_r(t) \in \mathbb{R}_{\sim}^{p}$, and $r << n$. . commensurating ROM (4) in first order generalized structure becomes:

$$\Upsilon_r\, q_r[k+1] = \alpha_r q_r[k] + B_{sr} u[k]$$

$$y_{sr}[k] = C_r q_r[k]$$

Remark: Equations (2) and (4) intimate that, as SOSS is transmuted in to identical 1^{st} order structure, the structure of matrices $\Upsilon, \alpha, B_s, C_s$ must be retained in ROM (4). If retention of structure does not occur, SOSS exposition go astray that relents imperceptive surmise of incipient dynamics. When curtailment takes place, ROM in (3) must not reflect haphazard pattern of entries of matrices of (1). The computational cost or storage limitations are very much there while handling system complexity in analysis alongwith design solicitations. This is due to the fact that number of state equations and state variables, for large scale systems (LSSs), are in billions. Accordingly, MOR contrivances are designed to surmise the characteristics of large system in resulting ROM.

II. RELATED WORK

Moore was pioneer with his concept of balanced truncation in 1981 [7] and this in fact is the most relevant paper of the domain under discussion. Nevertheless, method for model order reduction (MOR) was introduced by [8] in 1966 for generalized systems which was further modified by [9-10]. Structure-retaining extensions of classical model curtailment methods such as dominant pole contrivance, modal curtailment [11], moment-matching [12], equated curtailment [13], or for example of the H2-optimal iterative rational Krilov contrivance [14]. In [15], author introduced an equating concept that is strongly related to the origins of equated curtailment than the other extensions. In the context of first order systems, confabulations in [16], are aimed at such local surmises. Nevertheless, earlier curtailment approaches did not retain the structure in ROM [17-18]. But later developed MOR contrivances like modified Arnoldi, second-orderly structured equated truncation method (SOETM) and moment-matching (based on Krilov-subspaces) [2], [12], [15], [18-22] etc. retained second-orderly structure in ROM. In Krilov, Arnoldi, or moment-matching contrivances, although, curtailment of the model was achieved but ROM got unstable in the due course. Accordingly, a-priori curtailment error bounds do not exist. But stability of ROM is retained and so exist the a-priori curtailment error-bound for SOETM [17], [18], [23-25]. [13] portrayed multifarious SOETM contrivances for stable SOSSs.

Many day to day practical utilizations of physical systems demand aggrandization of ROM realization focused to intended frequency hiatus only. This guides to the chrysalis of limited frequency MOR (LFMOR) contrivances [5], [26][28]. In LFMOR, realization of ROM is focused over a predetermined hiatus of frequency . In LFMOR contrivances for stable systems bestowed so far, equating of controlability gramiann with observability gramiann over a predetermined frequency band is accomplished. The concept for LFMOR for standard systems was confabulated by [16] and same intellection was dilated by [29] for generalized systems. It is worth mentioning here the great contribution of Shafiq Haider, on the SOETM intellection. He discussed LFMOR of continuous and discrete time stable SOSSs using SOETM [26], [30]. Everything goes fine as long as the system is stable. The instability of the system poses a serious threat to the curtailment mechanism as the CALEs get unresolvable and the arithmetic of controlability and observability gramianns using CALEs is halted. Numerous MOR contrivances [31][40] etc. in this direction for sundry unstable systems have been endorsed. Before endorsing the curtailment mechanism, these contrivances stabilize the unstable system in a prior step. But irony of the situation is that, these contrivances do not bother about the issues faced when MOR of unstable SOSSs takes place. None of the aforecited contrivances contribute to second orderly structured forms of pristine unstable systems. Provision of structure-retention or physical exposition retention in ROM is not addressed at all. Eventually ROM generated through this carcass does not allow analysis and design process to be conducted and ROM becomes meaningless. Besides, LFMOR carcass, focussing on LF utilizations of unstable SOSSs, does not appear in literature.

Contemplating, the said research gap, in this publication, structure retaining MOR contrivances for LFMOR implementations of unstable SOSSs are initiated. To stabilize unstable SOSS, Bernouli stabilization feedback (BSF) and LFGs are elucidated by utilizing commensurating LF CALEs. The stabilized SOSS is supplicated in CALEs to procure gramianns for equated curtailment. A methodical and coherent scheme is invoked to procure Cholsky factorization of LFGs and solution of CALEs. Methodical gramianns are flaked in position and velocity snippets to ensure structure retention in ROM. Moreover, the procured gramianns are equated with different coalescences to find sundry SOETM contrivances. Curtailment relents the desired ROM that exhibit SOSS exposition as well as optimized accomplishment in intended frequency hiatus. Suggested contrivances are collocated with infinite gramianns contrivance for sundry unstable SOSSs to endorse the veracious development and dominance. Sequels for few examples (including one benchmark example from [41]) are depicted and accomplished juxtaposition of suggested contrivances is discussed. Manuscript is arranged as follows. Next section discusses preliminaries on stable and unstable SOSSs followed by suggested contrivances in section 4. Results are discussed in section 5 and conclusion is provided.

III. SECOND-ORDERLY STRUCTURED SYSTEMS

The transfer matrix of SOSS (1) is given by:

$$H(z) = (zC_{s2} + C_{s1})(z^2\mu_s + z\eta_s + \mathrm{K}_s)^{-1}B_{s2}$$

or $H = [\mu_s, \eta_s, \mathrm{K}_s, B_{s2}, C_{s1}, C_{s2}]$. System (1) is stable if no zeros of $P(\lambda) = \lambda^2\mu_s + \lambda\eta_s + \mathrm{K}_s$ lie outside the unit circle (all zeros might be inside the unit circle). Moreover, system (1) is controlable if
rank $[\lambda^2\mu_s + \lambda\eta_s + \mathrm{K}_s, B_{s2}] = n$ for all $\lambda \in C$
and is observable if
rank $[\lambda^2\mu_s^T + \lambda\eta_s^T + \mathbf{K}_s^T, \lambda C_{s2}^T + C_{s1}^T] = n$ for all $\lambda \in C$
commensuratingly, system (1) is controlable and observable if and only if the first order system (2) is controlable and observable respectively i.e. rank $[\lambda \curlyvee - \alpha, B] = 2n$ and rank $[\lambda \curlyvee^T - \alpha^T, C^T] = 2n$ for all $\lambda \in C$.
For stable system (1) (having all the eignvalues (EVs) of the pencil $\lambda \curlyvee - \alpha$ inside the unit-circle), the discrete gramianns would be:

$$W_c = \sum_{k=0}^{\infty} \phi_d^k B B^T (\phi_d^T)^k$$

$$W_o = \sum_{k=0}^{\infty} (\phi_d^T)^k C^T C \phi_d$$

are the positive semidefinite, unique and symmetric solutions of the DALEs:

$$\alpha W_c \alpha - \curlyvee^T W_c \curlyvee^T = -BB^T$$

$$\alpha^T W_o \alpha - \curlyvee^T W_o \curlyvee = -C^T C$$

where $\phi_d = e^{\curlyvee^{-1}\alpha k} \curlyvee^{-1}$ is the fundamental solution matrix of (2).
Nevertheless when SOSS of (1) is unstable, DALEs (8) and (9) provides improbable (indefinite) solution which consequently halts the sequence of equated curtailment. To avert this restraint, in the next section contrivances for MOR of unstable discrete time SOSs are suggested.

IV. METHODOLOGY

To avoid insolvability of DALEs (8) and (10) for unstable SOSs, in this section multiple non existing techniques for MOR of these systems are suggested.

A. PROPOSED TECHNIQUE-I (DBT)

In order to make DALEs solvable for unstable SOSs, large order system is decomposed in to stable and unstable portions and ROM of stable portion is computed. The obtained ROM is augmented with unstable portion to yield the required final ROM.

1 Decomposition in to stable and unstable portions:

The unstable discrete time SOS (2) can be written as (10).

$$H_t = [U^T \curlyvee U, U^T \alpha U, U^T B, CU, 0$$

$$= \left[\begin{bmatrix} \curlyvee_{t11} & \curlyvee_{t12} \\ 0 & \curlyvee_{t22} \end{bmatrix}, \begin{bmatrix} \alpha_{t11} & \alpha_{t12} \\ 0 & \alpha_{t22} \end{bmatrix}, \begin{bmatrix} B_{t1} \\ B_{t2} \end{bmatrix}, [C_{t1} \quad C_{t2}], 0\right]$$

In (10), the terms $\curlyvee_{t12}$ and α_{t12} represent the coupling between the states. Let $X_t = WX$ be a transformation that transform system in to decoupled form.

$$H_d = [W^{-1} \curlyvee_t W, W^{-1}\alpha_t W, W^{-1}B_t, C_t W, 0]$$

$$= \left[\begin{bmatrix} \curlyvee_{11} & 0 \\ 0 & \curlyvee_{22} \end{bmatrix}, \begin{bmatrix} \alpha_{11} & 0 \\ 0 & \alpha_{22} \end{bmatrix}, \begin{bmatrix} B_{1t} \\ B_t \end{bmatrix}, [C_{1t} \quad C_{2t}], 0\right]$$

The decomposition of system (11) in to stable and unstable portions yields (12).

$$H_d = H_s(\text{ Stable System }) + H_u(\text{ Unstable System })$$

$$= [\curlyvee_{11}, \alpha_{11}, B_{1d}, C_{1d}, 0] + [\curlyvee_{22}, \alpha_{22}, B_{2t}, C_{2t}, 0]$$

2) Reduction of stable portion

In reduction process of stable subsystem H_s, controllability and observability gramians W_{cs} and W_{os} are obtained for stable portion H_s by solving DALEs (8) and (10). To obtain the balanced transformation, consider the singular value decomposition of gramians:
$W_{cs} = U_p \Sigma_P V_p^T$ and $W_{os} = U_q \Sigma_q V_q^T$
and $V_L = U_q \Sigma_q{}^{-1/2}, V_R = U_P\sqrt{\Sigma_P}, \curlyvee = V_L^T V_R$. From these, BT is obtained as: $S_L = V_L U_\curlyvee V_\curlyvee{}^{-1/2}, S_R = V_R U_\curlyvee V_\curlyvee^{-1/2}$
The balanced system becomes:

$$H_{\mathrm{sbal}} = [\curlyvee_{bal}, \alpha_{bal}, B_{bal}, C_{bal}, 0]$$

$$= [S_L^T \curlyvee S_R, S_L^T \alpha S_R, S_L^T B, CS_R, 0]$$

$$H_{rs} = [\curlyvee_r, \alpha_r, B_r, C_r, 0]$$

The ROM for stable portion is obtained by truncating the above balanced system to required order.

3) Overall Reduced Model for unstable Discrete Time SOS

Reduced stable and decomposed unstable portions are combined to obtain the required ROM.
$H_r = H_{rs} + H_u$
Remark : The ROM (13) obtained from suggested technique DBT does not provide structure preservation. Moreover addition of unstable dynamics in ROM also degrade ROM performance. Also if unstable part of system is of large order, it becomes impossible to avoid large order ROM, because of augmentation of unstable portion.
In order to avoid above restraints, in next subsection, structure preserving as well as stabilized dynamics of pristine system has been invoked in reduction process.

B. SUGGESTED TECHNIQUE II

In order to make DALEs solvable for unstable discrete time SOS (2) Bernoulli feebback stabilization is solicited to stabilize the system and stabilized system is used to compute gramianns. Thereafter gramianns are fragmented in to position and velocity snippets to acquire structure retention. Next, ROM is formulated using discrete time second order equated truncation

(DSOET). 1) Stabilizing Solution for Unstable Discrete Time SOS In a foremost step, Bernouli feedback solution gramianns are formulated from (14)-(17), to stabilize system (2), [42]

$$\alpha W_c \alpha - \gamma^T W_c \gamma^T = -BB^T$$

$$\alpha^T W_o \alpha - \gamma^T W_o \curlyvee = -C^T C$$

where $K_c = B^T X_c \curlyvee$ and $K_o = \curlyvee X_o C^T$ are known as Bernoulli stabilizing feedback matrices, due to the fact that the matrices X_c and X_o are the respective stabilizing solutions of the generalized algebraic Bernoulli equations (16) and (17)

$$\alpha^T X_c \alpha - \curlyvee^T X_c \curlyvee = \curlyvee^T X_c BB^T X_c \curlyvee$$

$$\alpha X_o \alpha^T - \curlyvee X_o \curlyvee^T = \curlyvee X_o C^T C X_o \curlyvee^T$$

Partitioning of gramianns obtained from (14) and (15) yields

$$W_{cstb} = \begin{bmatrix} W_{pcstb} & W_{12cstb} \\ W_{12cstb}^T & W_{vcstb} \end{bmatrix},$$

$$W_{ostb} = \begin{bmatrix} W_{\text{postb}} & W_{12ostb} \\ W_{12ostb}^T & W_{\text{vostb}} \end{bmatrix},$$

where $W_{vostb}, W_{\text{postb}}, W_{vcstb}$ and W_{pcstb} are velocity and position observability and controllability gramianns respectively.

Remark: The gramianns are computed from (14) and (15) using stabilized system in suggested contrivance-II. Moreover, partitioning of gramianns in to position and velocity portions, ensures the structure preservation in ROM. Also, in suggested contrivance-II, there is no need to augment any large order unstable portion. Consideration of these aspects, ensures a better and meaningful surmise of incipient system.

PERFORMANCE MEASURE

Before discussing the model reduction, it seems rather realistic to introduce the performance measure applied in the entire reduction process. In model reduction, error performance measure is the difference between pristine model and resulting curtailed model. It can be formulated by using norms say in [37] or by using average error as in [26]. The absolute value of average error is formulated as:

$$e_k = \left| \sum_{i=1}^{N} \left(\frac{y_{ki} - y_{kri}}{N} \right) \right|$$

where, $k = 1 \ldots p, i = 1 \ldots N$

$N =$ number of output data points in considered hiatus of time using corresponding contrivances mentioned in the algorithms.

The product of position and velocity gramianns yields following definition.

Definition 1: Consider a stablized system as in equation (1) [13] 1) The square root of the eigenvalues of product $W_{\text{pcstb}} W_{\text{postb}}$ are the position HSVs.

2 The square root of eigenvalues of product $W_{vcstb} M^T W_{vostb} M$ are the velocity HSVs.

3 The square root of eigenvalues of product $W_{pcstb} M^T W_{vostb} M$ are the position-velocity HSVs.

4 The square root of eigenvalues of product $W_{vcstb} W_{\text{postb}}$ are the velocity-position HSVs.

EQUATED TRANSFORMATION FOR STABILIZED SOS

For second-orderly equated transformation, different equating contrivances can be solicited to stabilized system in order to obtain velocity and position HSVs.

Definition 2: The stabilized SOS is called:

1 Position equated if $W_{pcstb} = W_{postb} = \text{diag}\left(\zeta_1^{pstb}, \ldots, \zeta_n^{pstb}\right)$

2 Velocity equated if $W_{vcstb} = W_{vostb} = \text{diag}\left(\zeta_1^{vstb}, \ldots, \zeta_n^{vstb}\right)$

3 Position-velocity equated if $W_{pcstb} = W_{vostb} = \text{diag}\left(\zeta_1^{pvstb}, \ldots, \zeta_n^{pvstb}\right)$.

4 Velocity-position equated if $W_{vcstb} = W_{\text{postb}} = \text{diag}\left(\zeta_1^{vpstb}, \ldots, \zeta_n^{vpstb}\right)$.

where ζ represents either velocity or position or both velocity-position (position-velocity) HSVs arranged in descending order. Equated transformation can be derived by considering the Cholsky factorization of gramianns given as:

$$W_{pcstb} = R_{pstb} R_{pstb}^T \quad W_{vcstb} = R_{vstb} R_{vstb}^T$$

$$W_{\text{postb}} = L_{pstb} L_{pstb}^T \quad W_{vostb} = L_{vstb} L_{vstb}^T$$

where R_{pstb}, R_{vstb}, L_{pstb}, $L_{vstb} \in R^{n\times n}$ are lower triangular nonsingular Cholsky factors which are used to find out HSVs via classical singular values as given below:

$$\left(\zeta_i^{pstb}\right)^2 = \lambda_i\left(W_{pcstb} W_{postb}\right)$$

$$= \lambda_i\left(R_{pstb} R_{pstb}^T L_{pstb} L_{pstb}^T\right)$$

$$= \lambda_i\left(L_{pstb}^T R_{pstb} R_{pstb}^T L_{pstb}\right) = \sigma_i^2\left(R_{pstb} L_{pstb}\right)$$

where σ_{istb} 's are classical SVs. Similarly $\zeta_i^{vstb} = \sigma_i(R_{vstb}^T M^T L_{vstb}), \zeta_i^{pvstb} = \sigma_i\left(R_{pstb}^T M^T L_{vstb}\right)$, and $\zeta_i^{vpstb} = \sigma_i\left(R_{vstb}^T L_{pstb}\right)$. Calculating the singular value decomposition of these products provide:

$$R_{pstb}^T L_{pstb} = U_{pstb} \Sigma_{pstb} V_{pstb}^T$$

$$R_{vstb}^T M^T L_{vstb} = U_{vstb} \Sigma_{vstb} V_{vstb}^T$$

$$R_{pstb}^T M^T L_{vstb} = U_{pvstb} \Sigma_{pvstb} V_{pvstb}^T$$

$$R_{vstb}^T L_{pstb} = U_{vpstb} \Sigma_{vpstb} V_{vpstb}^T$$

where $U_{pstb}, V_{pstb}, U_{vstb}, V_{vstb}, U_{pvstb}, V_{pvstb}, U_{vpstb}, V_{vpstb}$ are the orthogonal matrices while:

$$\Sigma_{pstb} = \text{diag}\left(\zeta_1^{pstb}, \ldots, \zeta_n^{pstb}\right)$$

$$\Sigma_{vstb} = \text{diag}\left(\zeta_1^{vstb}, \ldots, \zeta_n^{vstb}\right)$$

$$\Sigma_{pvstb} = \text{diag}\left(\zeta_1^{pvstb}, \ldots, \zeta_n^{pvstb}\right)$$

$$\Sigma_{vpstb} = \text{diag}\left(\zeta_1^{vpstb}, \ldots, \zeta_n^{vpstb}\right)$$

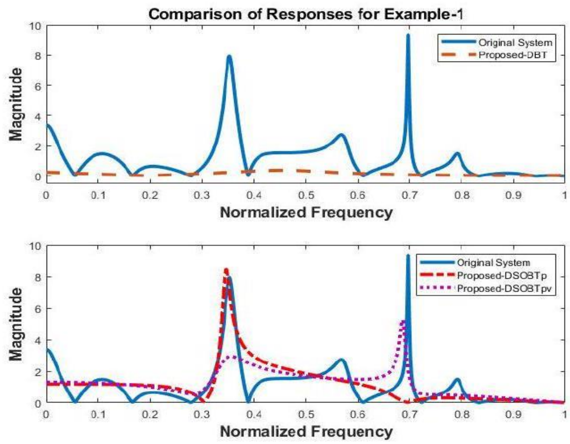


FIGURE 1: juxtaposition of ripostes for Example-1

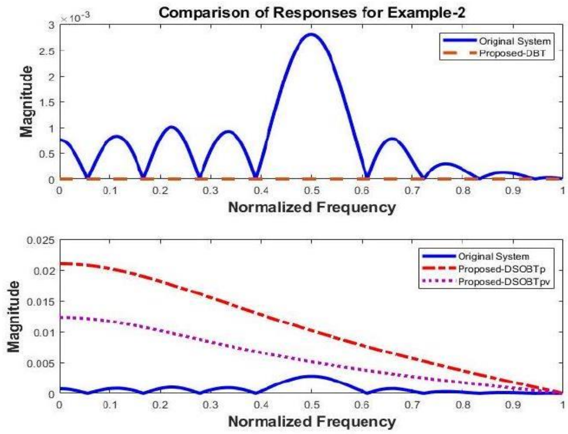


FIGURE 2: juxtaposition of ripostes for Example-2

are nonsingular matrices. Relationship in equation (20) can be utilized to find out the equated transformation (P_{lstb}, P_{rstb}) for aforementioned contrivances of definition 2.

1 Equated Truncation for Stabilized Discrete Time SOS

As mentioned in the previous section, the equated transformation yields HSVs. The magnitudes of these HSVs show the extent to which the velocity and position states are involved in the dynamics of the system. States corresponding to small magnitudes have minute involvement and are truncated at small cost of reduction error. Two algorithms for velocity and position equated curtailment are presented here for stabilized discrete time SOSs.

Algorithm 1: Position Balancing based Discrete Time Limited Frequency SOBT (LFDSOBTp)

Input: Given a stable SOSS

$H = [\mu_s, \eta_s,\ \mathrm{K}_s, B_{s2}, C_{s1}, C_{s2}]$ and frequency hiatus $\hat{\Delta}$:

Output: The FLROM

$H_r = [\mu_{sr}, \eta_{sr},\ \mathrm{K}_{sr}, B_{s2r}, C_{s1r}, C_{s2r}]$ 1. Calculate the DLFGs $W_{\hat{p}c\hat{\Delta}}, W_{\tilde{p}o\hat{\Delta}}, W_{\hat{v}c\hat{\Delta}}$ and $W_{\tilde{v}o\hat{\Delta}}$ using (14) and (15).

2 Calculate the Cholsky factors $R_{p\hat{\Delta}}, R_{v\hat{\Delta}}, L_{p\hat{\Delta}}$ and $L_{v\hat{\Delta}}$

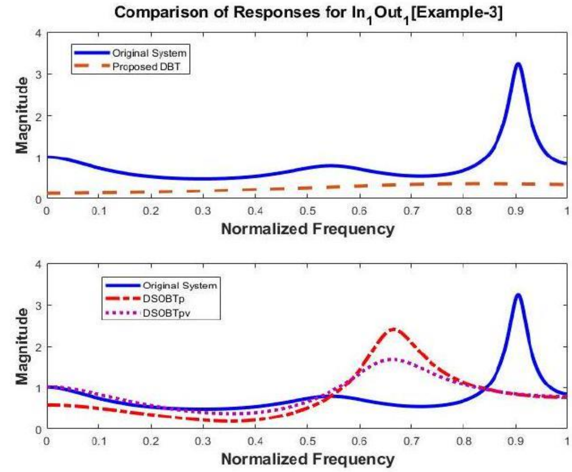


FIGURE 3: juxtaposition of ripostes for $I_1 - O_1$ Example-3

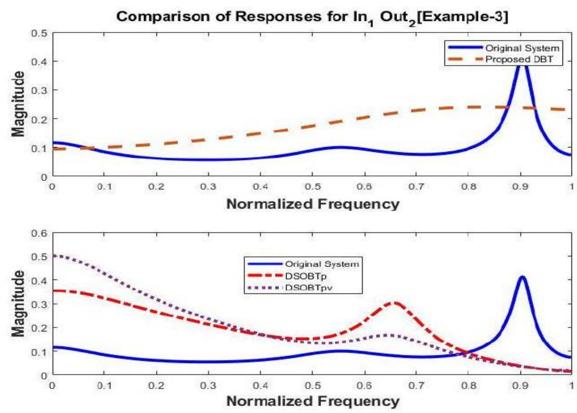


FIGURE 4: juxtaposition of ripostes for $I_1 - O_2$ Example-3

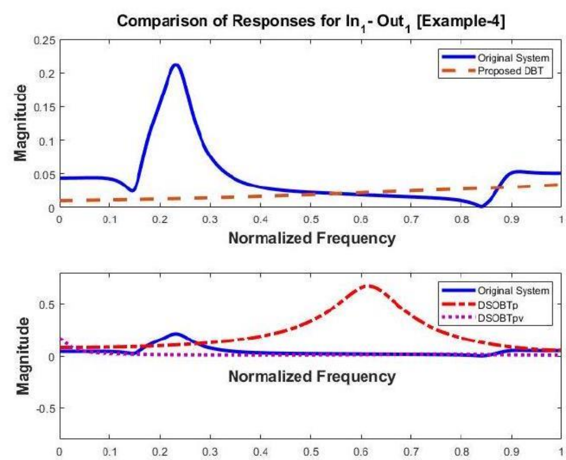


FIGURE 5: juxtaposition of ripostes for $I_1 - O_1$ Example-4

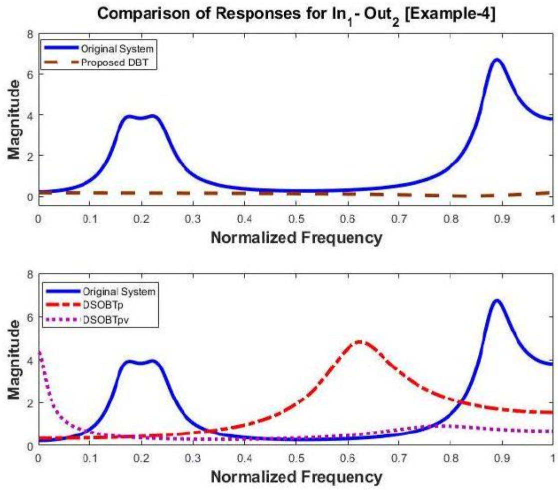
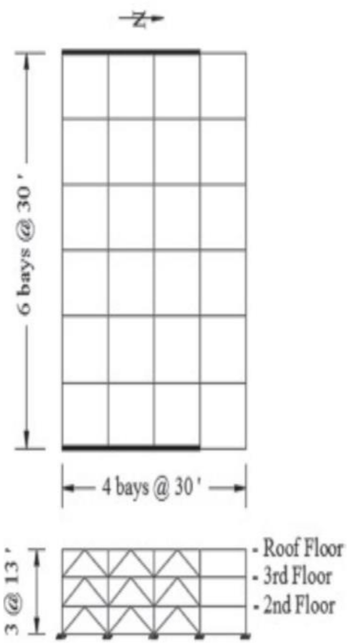


FIGURE 6: juxtaposition of ripostes for $I_1 - O_2$ Example-4



Note: 1ft = 0.3 m

FIGURE 7: Three story model building

$$R_{v\Delta}^T M^T L_{v\hat{\Delta}} = [U_{v1\hat{\Delta}} \quad U_{v2\hat{\Delta}}] \begin{bmatrix} \tilde{\Sigma}_{v1\hat{\Delta}} & 0 \\ 0 & \tilde{\Sigma}_{v2\hat{\Delta}} \end{bmatrix} [V_{v1\hat{\Delta}} \quad V_{v2\hat{\Delta}}]^T$$

$\tilde{\Sigma}_{p1\hat{\Delta}} = \text{diagnn}\left(\hat{\zeta}_1^{p\hat{\Delta}}, \dots, \hat{\zeta}_r^{p\hat{\Delta}}\right)$ $\tilde{\Sigma}_{p2\hat{\Delta}} =$ diagnn $\left(\hat{\zeta}_{r+1}^{p\hat{\Delta}}, \dots, \hat{\zeta}_n^{p\hat{\Delta}} 115\right)$ are symmetric, unique and positive definite solutions of

4 Calculate the ROM

$\mu_{sr} = P_{l\hat{\Delta}}^T M P_{r\hat{\Delta}}$ $\eta_{sr} = P_{l\hat{\Delta}}^T D P_{r\hat{\Delta}}$ $\mathrm{K}_{sr} = P_{l\hat{\Delta}}^T K P_{r\hat{\Delta}}$ $B_{s2r} = P_{l\hat{\Delta}}^T B_{s2}$ $C_{s1r} = C_1 P_{r\hat{\Delta}}$ $C_{s2r} = C_2 P_{r\hat{\Delta}}$ where $P_{l\hat{\Delta}} = L_{v\hat{\Delta}} V_{v1\hat{\Delta}} \tilde{\Sigma}_{p1\hat{\Delta}}^{-1/2}$ and $P_{r\hat{\Delta}} = R_{p\hat{\Delta}} U_{p1\hat{\Delta}} \tilde{\Sigma}_{p1\hat{\Delta}}^{-1/2}$

Algorithm 2: Position-Velocity Balanced Discrete Time Limited Frequency SOBT (LFDSOBTpv)

Input: Given a stable SOSS

$H = [\mu_s, \eta_s, \mathrm{K}_s, B_{s2}, C_{s1}, C_{s2}]$ and frequency hiatus $\hat{\Delta}$:

Output: The FLROM

$H_r = [\mu_{sr}, \eta_{sr}, \mathrm{K}_{sr}, B_{s2r}, C_{s1r}, C_{s2r}]$ 1. Calculate the DLFGs $W_{\hat{p}c\hat{\Delta}}$ and $W_{\hat{v}o\hat{\Delta}}$ using (14) and (15).

2 Calculate the Cholsky factors $R_{p\hat{\Delta}}$ and $L_{v\hat{\Delta}}$ of DLFGs $W_{\hat{p}c\hat{\Delta}}$ and $W_{\hat{v}o\hat{\Delta}}$ as given in (19).

3 Compute SVD for the products:

$$R_{p\hat{\Delta}}^T M^T L_{v\hat{\Delta}} = [U_{pv1\hat{\Delta}} \quad U_{pv2\hat{\Delta}}] \begin{bmatrix} \tilde{\Sigma}_{\hat{p}v1\hat{\Delta}} & 0 \\ 0 & \tilde{\Sigma}_{\hat{p}v2\hat{\Delta}} \end{bmatrix} [V_{\hat{p}v1\hat{\Delta}}$$

where $[U_{\hat{p}v1\hat{\Delta}} \quad U_{\hat{p}v2\hat{\Delta}}]$ and $[V_{\hat{p}1\hat{\Delta}} \quad V_{\hat{p}v2\hat{\Delta}}]$ are orthogonal and

$\tilde{\Sigma}_{\hat{p}1\hat{\Delta}} = \text{diagnn}\left(\hat{\zeta}_1^{\hat{p}\hat{v}\hat{\Delta}}, \dots, \hat{\zeta}_r^{\hat{p}\hat{v}\hat{\Delta}}\right)$, $\tilde{\Sigma}_{\hat{p}2\hat{\Delta}} =$ diagnn $\left(\hat{\zeta}_{r+1}^{pv\hat{\Delta}},\right.$

4 Calculate the ROM

$$\mu_{sr} = I_n \qquad \eta_{sr} = P_{l\hat{\Delta}}^T D P_{r\hat{\Delta}} \quad K_r = P_{l\hat{\Delta}}^T K P_{r\hat{\Delta}}$$
$$B_{s2r} = P_{l\hat{\Delta}}^T B_{s2} \quad C_{s1r} = C_1 P_{r\hat{\Delta}} \quad C_{s2r} = C_2 P_{r\hat{\Delta}}$$

V. RESULTS AND DISCUSSION

The suggested MOR contrivances are solicited to sundry unstable SOSs like piezoelectric system and some of the self generated examples (given in appendix) and their results are presented. A 3-story model building retrofitted with piezoelectric braces in downtown Los Angeles [43] is shown in Figure 5 and coefficient matrices:

$$\mu_s = \begin{bmatrix} 9.3 & 0 & 0 \\ 0 & 6.3 & 0 \\ 0 & 0 & 4.7 \end{bmatrix}, \eta_s = \begin{bmatrix} 7.4 & 2 & 7 \\ 5 & 6.1 & 0 \\ 1 & 3 & 2.1 \end{bmatrix}$$
$$\mathrm{K}_s = \begin{bmatrix} 8.8 & 3.7 & 0 \\ -2.3 & 8 & 1 \\ -3 & 0 & 7.6 \end{bmatrix}$$

In our work we diagonally extend the three story structure to twelse story structure having three bunches of identical pîêzêelectric elements. The 12th order system is curtailed to $r = 5$ utilizing suggested infinite hiatus contrivances. As all the eignvalues for this model are unstable; therefore, it becomes imppracticable to break system in to stable and apasisaple portions for suggested technique-I. On the other hand, suggested contrivance II depicts an overwhelming riposte for ROM in position as well as position velocity equating as portrayed in Figure 2. Figure 2 depicts riposte for position and position velocity balancing on infinite range of frequency.

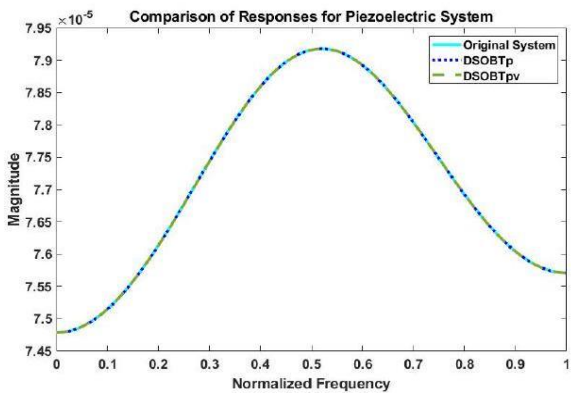


FIGURE 8: Comparison of ripostes for piezoelectric system

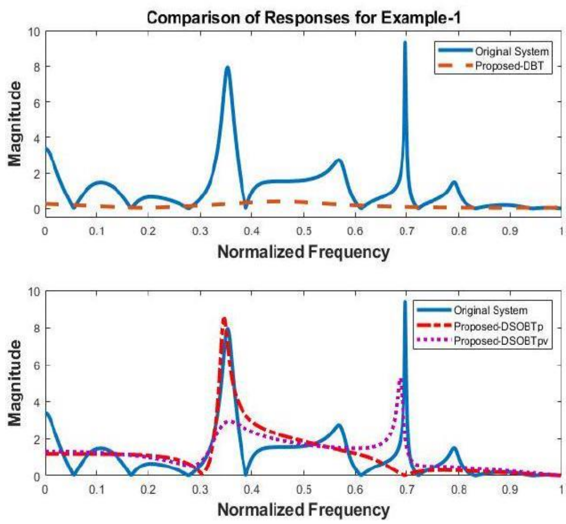


FIGURE 9: Comparison of ripostes for Example-1

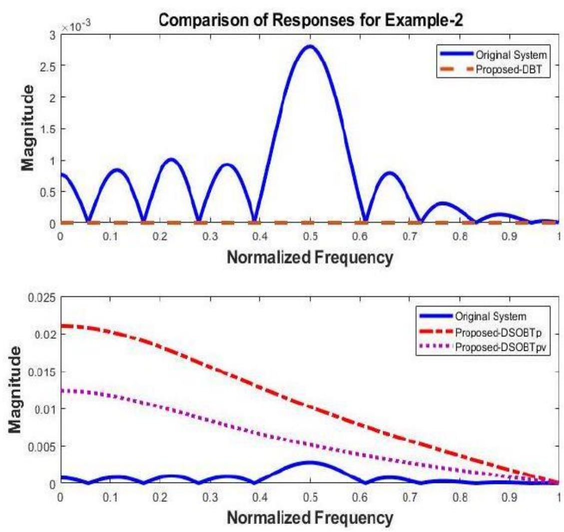


FIGURE 10: Comparison of ripostes for Example-2

Stating once more that second contrivance retains second-orderly structure and exhibit stabilized system dynamics, therefore, this contrivance seems more promising. SISO Systems: The 9th order system in Example-1 is a single input single output (SISO) system. This system is curtailed to $r = 3$ using suggested technique-I as well as suggested DSOETp and DSOETpv contrivances as imparted in algorithm 1 and 2 . The ripostes of pristine and curtailed systems have been depicted in Figure 3. It is noticed that the riposte of suggested technique-I is worst among the suggested contrivances as it does not cater for second-orderly structure as well as bring in unstable dynamics in ROM. Do consider that the curtailed model of DSOETp and DSOETpv contrivances very closely surmise the pristine SOSS.

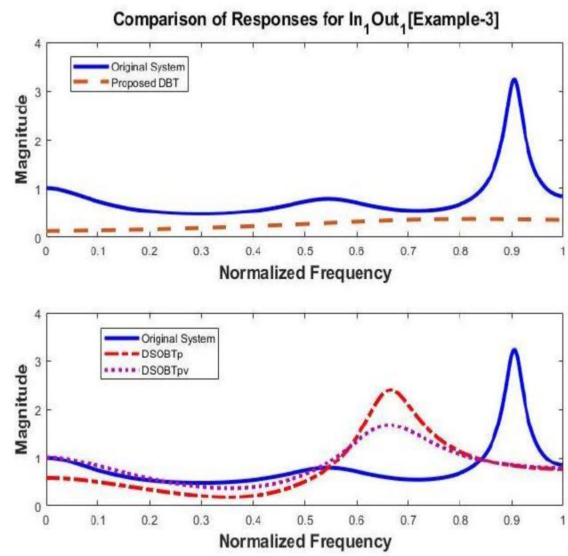


FIGURE 11: Comparison of ripostes for $I_1 - O_1$ Example-3

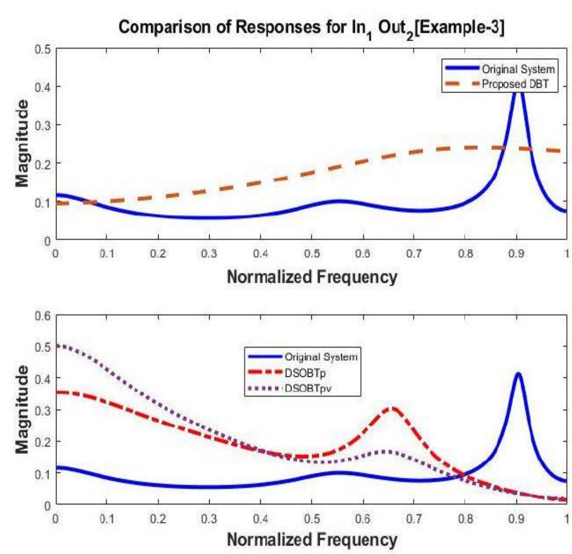


FIGURE 12: Comparison of ripostes for $I_1 - O_2$ Example-3

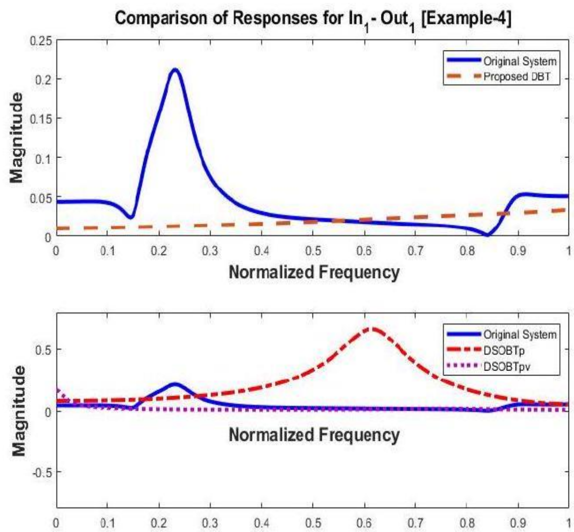


FIGURE 13: Comparison of ripostes for $I_1 - O_1$ Example-4

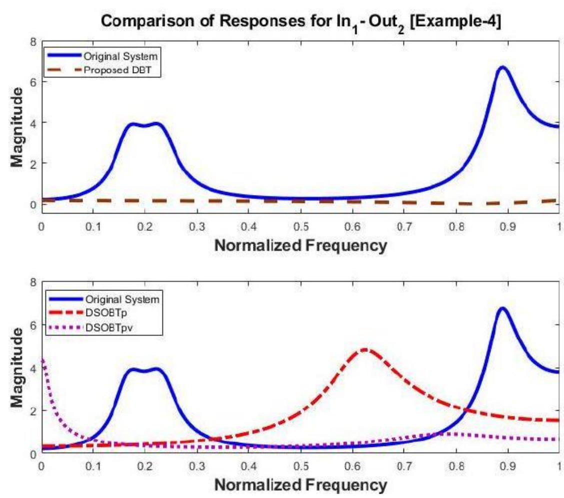


FIGURE 14: Comparison of ripostes for $I_1 - O_2$ Example-4

Further, 9th order SISO SOSS of Example-2 is curtailed to $r = 1$ and ripostes for pristine and curtailed systems are portrayed in Figure 4. The reason for curtailing to $r = 1$ was due to the fact that system has only one eignvalue in unit circle and suggested technique-I could only be solicited for r = 1. This indicates one of the drawbacks of the suggested technique-I. Aditionally, it is again opined that the riposte of suggested technique-I is not up to the mark among the suggested contrivances while ripostes for ROM procured from DSOETp and DSOETpv contrivances depicts a very close surmise of pristine system riposte.

MIMO Systems: The 10th order MIMO system of Example3 with single input and two outputs is curtailed to $r = 5$ and riposte for pristine and ROMs utilizing suggested techniqueI and that of contrivances DSOETp and DSOETpv are depicted in Figure 5 and Figure 6. It is again quite vivid in these depictions that the riposte of suggested technique-I is worst among the suggested contrivances. The ROMs closely surmise pristine large order MIMO system of Example 3 for the suggested contrivances of DSOETp and DSOETpv.

Finally, the 8th order MIMO system of Example-4 with single input and two outputs is curtailed to $r = 4$ and riposte for pristine and ROMs procured using the suggested contrivances are shown in Figure 7 and Figure 8. This exemplification depicts non compliance of the riposte of suggested technique-I among the suggested contrivances once more.

VI. CONCLUSION

Sundry non-existent model order reduction (MOR) contrivances for discrete time unstable second-orderly structured systems are suggested. These contrivances cater for inresolvability of DALEs for unstable discrete time SOSSs. In first proposition unstable SOSS is broken in to stable and unstable snippets and equated curtailment paradigm is solicited to stable portion. The obtained ROM for stable portion is augmented with unstable portion to procure the overall curtailed system. While in second proposition, system is first stabilized using Bernouli feedback stabilization carcass and then gramianns are formulated for stabilized system. Further gramianns are fragmented in to position and velocity snippets to acquire structure preservation in ROM and equated curtailment gets applied. It can be concluded that second-orderly structure in ROM for first contrivance gets lost as well as augmented unstable dynamics degrade the ROM performance. But second contrivance retains second-orderly structure as well as involves stabilized system dynamics, therefore, this contrivance seems more convincing. Suggested contrivances are tested on multiple unstable SOSs and sequels for few SISO and MIMO systems are presented. The results certify the veracious development and preeminance of the later suggested contrivances.

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APPENDIX

Different unstable discrete time SISO and MIMO SOSs Example 1: A 9th order SISO SOS

$$K=\begin{bmatrix} -1.0 & 0.4 & 0.2 & 1.01 & 1.0 & 0.4 & 0.2 & 1.0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\eta=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1.7 & 0.6 & 0.1 & 0.3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\beta_2 = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T, C_1$
$= [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1$
$C_2 = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1], \mu = I_9$

Example 2: A 9^{th} order SISO SOS

$$K=\begin{bmatrix} -0.8 & -1.2 & -1 & 1 & -0.7 & -8.4 & 3.4 & -12.6 & -1.2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\eta=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1], \mu = I_9$
$\beta_2 = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T,$ $$
$C_1 = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]$
C_{1}=[1-1-1-1-1]
$$

$$K=\begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.1 & -8 & 0.1 & 0 & 0 & -115 & 0.1 \\ 0.1 & 0 & 0 & 0 & 0.1 & -8 & 0.1 & 0 & 0.2 & 0.1 \\ 0.1 & 0.1 & 1.1 & 0 & 0.9 & 0.1 & -8 & 0.1 & 0 & 0 \\ 0.1 & 0.1 & -8 & 0.1 & 0.1 & -8 & 0.1 & 0 & 0 & 0.2 \\ 0.1 & 0.1 & 0.2 & 0.1 & -8 & 0.1 & 0 & 0.1 & 0 & 0.1 \\ 0.1 & 0.2 & 0 & 0.1 & -8 & 0.1 & 0 & 0.2 & 0.1 & 0.2 \\ 0.1 & 0 & 1 & 0 & 0.1 & -8 & 0.1 & 0 & 0.2 & 0.1 \\ 0.1 & 0 & 0 & 0 & 0.1 & -8 & 0.1 & 0 & 0.2 & 0.1 \\ 0.1 & 0 & 0 & 0 & 0.1 & -8 & 0.1 & 0 & 0.2 & 0.1 \\ 0.1 & 0 & 0 & 0 & 0.1 & -8 & 0.1 & 0 & 0.2 & 0.1 \end{bmatrix};$$

$$\eta=\begin{bmatrix} 0.8 & 0.2 & 0.1 & 0.2 & 0.8 & 0.2 & 0.1 & 0.2 & 0.95 & 0.2 \\ 0.1 & 0.2 & 0.8 & 0.8 & 0.2 & 0.1 & 0.2 & 0.2 & 0.1 & 0 \\ 0.8 & 0.2 & 0.1 & 0.2 & 0.8 & 0.2 & 0.1 & 0.2 & 0.5 & 0.2 \\ 0.1 & 0.5 & 0.8 & 0.2 & 0.1 & 0.2 & 0.1 & 0.1 & 0.2 & 0.8 \\ 0.8 & 0.2 & 0.1 & 0.2 & 0.8 & 0.2 & 0.1 & 0.2 & 0.5 & 0.2 \\ 0.2 & 0.8 & 0.2 & 0.8 & 0 & 0.8 & 0.2 & 0.1 & 0.2 & 0.8 \\ 0.8 & 0.2 & 0.1 & 0.2 & 0.8 & 0.2 & 0.1 & 0.2 & 0.5 & 0.2 \\ 0.1 & 0.3 & 0.1 & 0.2 & 0.8 & 0.2 & 0.1 & 0.2 & 0.3 & 0.2 \\ 0.8 & 0.2 & 0.1 & 0.2 & 0.8 & 0.2 & 0.1 & 0.2 & 0.5 & 0.2 \\ 0.8 & 0.8 & 0.2 & 0.1 & 0.2 & 0.2 & 0.2 & 0.3 & 0.2 & 0.2 \end{bmatrix}$$

$$\beta_2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^T, C_1=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C_2=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \mu = I_{10}$$

Example 4: A 8th order MIMO SOS

$$K=\begin{bmatrix} 0.12 & 0.11 & -28 & 0.12 & 1.12 & 0.19 & 0.9 & 0.11 \\ 1.12 & 0.01 & -117 & 0.11 & 0.12 & 0.40 & 0.21 & 0.12 \\ 1.12 & 0.01 & -16 & 0.11 & -18 & 0.12 & -115 & 0.12 \\ -18 & 0.12 & 0.11 & -105 & 0.12 & 0.4 & 0.04 & 0.2 \\ 0.2 & 0.9897 & -118 & 0.76 & 0.04 & 0.12 & 0.03 & 0.12 \\ 0.01 & 0.11 & -18 & 0.1866 & 0.04 & 0.299 & 0.1779 & 0.21 \\ 0.111 & 0.044 & 0.11 & -28 & 0.12 & 0.04 & 0.21 & 0.12 \\ 0.019 & 0.04 & 0.11 & -118 & 0.12 & 0.04 & 0.21 & 0.12 \end{bmatrix};$$

$$\eta=\begin{bmatrix} 0.08 & 0.3322 & 0.11 & -88 & 0.08 & 0.22 & 0.911 & 0.82 \\ 0.021 & 0.12 & 0.08 & -8 & 0.22 & 0.11 & 0.82 & 0.02 \\ 0.08 & -57 & 0.11 & 0.82 & 0.08 & 0.22 & 0.11 & 0.82 \\ 0.01 & 0.15 & -8 & 0.22 & 0.11 & 0.82 & 0.0189 & 0.31 \\ 0.08 & 0.22 & 0.11 & 0.82 & 0.08 & 0.22 & 0.1778 & 0.82 \\ 0.32 & 0.18 & 0.22 & 0.34 & 0 & 0.056 & 0.22 & 0.11 \\ 0.08 & 0.22 & 0.11 & 0.82 & 0.54 & 0.22 & 0.12 & 0.82 \\ 0.31 & 0.131 & 0.77 & 0.12 & 0.08 & 0.22 & 0.11 & 0.12 \end{bmatrix};$$

$$\beta_2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^T$$

$$C_1=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_2=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \mu = I_8$$