Fitting Distributions to The Annual Maximum Rainfall in Punjab and Azad Jammu and Kashmir

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Abstract- This study analyzed the fit of two-parameter Frechet, Weibull, log-logistic, and gamma distribution to annual maximum rainfall in eight different sites of Punjab and Azad Kashmir, from the period 1980-2015. The maximum likelihood method, Moments, Bayesian and probability-weighted moment methods are used for the estimation of parameters and the Kolmogorov Smirnove has been applied for the goodness of fit criterion. Based on scores of the goodness of fit test, candidate distributions are found the best fit for annual maximum rainfall data for selected sites of Punjab and Azad Jammu and Kashmir. Also, the graphs of probability density functions are constructed for comparison purposes amongst the estimation methods. Moreover, rainfall quantiles with the best-fitted method (for many return periods) of each site have been estimated using the quantile functions of the candidate distributions for future prediction.

Keywords- Weibull Distribution, Frechet Distribution, Gamma Distribution, Log Logistic Distribution, Maximum Likelihood Method, Probability Weighted Moment Method, Moment Estimation Method, Bayesian Estimation Method.

I. INTRODUCTION

The weather has been divided into two major categories hot and cold. These categories of weather prevail in the world. However, Pakistan is one of those fortunate pieces of land in which nature has blessed diversity of weather patterns such as spring, summer and autumn, under the great intensity of hotness and some are under the shiver of cold. An appropriate climate is very indispensable for agriculture. Rainfall is mandatory because it has negative and positive impacts on Pakistan soil and profuse rainfall is pernicious for Pakistan in terms of floods and can bring about uncalculated destruction and unlimited loss of lives.

A regional study is important to frame the extreme behaviour of rainfall. Several methods have been suggested for modelling rainfall extents. For instance, a study made by [1] on exponential distributions that explain the rainfall behaviour in Peninsular Malaysia over a multiyear period and suggested that a combination of two distributions was better rather than a single distribution for explaining the regular rainfall amount. Moreover, the applied generalized extreme value (GEV) distribution explains the intensity of rainfall and to give an idea of its future behaviour studied by [2]. Similarly, [3-4] used different distributions such as beta and gamma distributions. Also [5] used GEV and generalized Pareto (GP) distributions that model the rainfall pattern in Malaysia and got the result that GP and GEV models were suitable.

Pakistan is a land of agriculture segment and rigid climate can be determinate for agriculture. An appropriate climate is very indispensable for agriculture. Rainfall is Mandatory because it has negative and positive impacts for Pakistan soil and profuse rainfall is pernicious for Pakistan in term of flood and can bring about uncalculated destruction and unlimited loss of lives.

According to statistical point of view, the regional study is important frame for the extreme behavior of rainfall. For several sites, regional rainfall frequency analysis has been used annual maximum rainfall data. The analysis involves reviewing the sites, checking if the expected areas are really the same and select the appropriate distribution to fit the region's data.

1.1. Rainfall in Punjab

Punjab is considered most populated province of Pakistan and it is also the second largest regarding area and 56% population of Pakistan lives in Punjab. Punjab has been separated into two major categories, southern Punjab, and northern Punjab, in terms of rainfall pattern. The Southern part of the province receives less rainfall as compared to the northern part of the province. The rainy period in

Punjab happening from July to September and shown that Average annual rainfall range between 96 cm in the mountain regions and 46 cm in the plains regions.

1.2. Rainfall in Azad Jammu and Kashmir (AJ&K)

The State of AJ&K is situated in mostly mountainous deeply classified by rugged, valleys and irregular terrain. AJ&K has the extensive range of climatic circumstances that depends on the altitude, ranges from 6,325 meters in the North to 360 meters in the South. It is found that, throughout most of the regions of this State, the average rainfall is more than 1400 millimeters (mm) and the highest rainfall are recorded near Muzaffarabad which is more than 1800mm during the summer period. Moonsoons flood in river Neelum and Jhelum is common due to extreme rain and snow melting.

For annual maximum rainfall analysis (AMRFA), several theoretical distributions are being used. In this study Frechet distribution (FD), Weibull distribution (WD), log-logistic distribution (LLD) and gamma distributions (GD) with four estimation methods, maximum likelihood (ML), Bayesian estimation (BE), probability weighted moment (PWM) and moment estimation (ME) are used to model the AMRF data of eight sites, namely; Lahore, Islamabad, Murree, Jhelum, Mianwali, Muzaffarabad, Sialkot, and Kotli.

II. LITERATURE REVIEW

Several probability distributions with estimations methods have been suitable in the literature for modelling rainfall data across the world including Pakistan. Certain published native and international studies are given below.

2.1. International Studies and Case

A statistical assessment of rainfall intensity and frequency, taking into account the timing of rainfall within the daily cycle conducted by [6]. Monthly precipitation records for six meteorological regions of the Turkish Republic of Northern Cyprus (TRNC) from September 1975 to August 2014 were analyzed by [7]. To evaluate rainfall behavior over the past 20 years in Solapur district, annual rainfall data from 2000 to 2019 was utilized in [8]. In their study, [9] examined trends in rainfall and discussed various techniques for detecting these patterns. The Mann-Kendall non-parametric test was applied for identifying trends, while Sen's slope estimator helped determine the rate of change. Their findings highlighted that rainfall trends differ by region and are influenced by geographical features such as topography. [10] explored how machine learning (ML) algorithms can be employed to model and forecast rainfall in Sokoto, Nigeria. Their study tested four ML approaches—Linear Discriminant Analysis (LDA), Support Vector Machine (SVM), K-Nearest Neighbors (KNN), and Naïve Bayes (NB)—using historical climate data. In [11], the researchers focused on seasonal flood risks by analyzing rainfall during the monsoon (June–September) and post-monsoon (October–December) months. Their findings contribute to better insights into the flood risks associated with these periods under varying return intervals.

A thorough framework was adopted in [12] to evaluate rainfall probability distributions throughout Pakistan. This approach aimed to produce a probabilistic zoning map to support effective water resource planning and flood risk mitigation across different climates and landscapes. Rainfall data from the Pakistan Meteorological Department, collected over 42 years, was validated using CHIRPS datasets. The objective of [13] was to deepen the understanding of hydroclimatic risks and their temporal interconnections in the Upper Jhelum Basin (UJB) in South Asia, considering both historical and future climate scenarios. The research proposes alternative water management practices that are affordable, practical, and environmentally sustainable. In [14], long-term spatial and temporal variations in rainfall across India were studied using data from 1901 to 2015 at the meteorological division level. The Pettitt test identified significant change points in the data series, while the Mann-Kendall test and Sen's trend analysis were used to assess long-term patterns. The findings have implications for sustainable water resource management in the face of limited availability. According to [15], global patterns in rainfall trends and extremes do not consistently reflect a single dominant cause such as increased carbon dioxide emissions. This underscores the importance of stochastic modeling over deterministic approaches when predicting future hydroclimatic behavior. [16] warned of a concerning outlook involving increased heatwaves and agricultural drought, emphasizing the urgent need for adaptive and preventive strategies. Meanwhile, [17] demonstrated that the Holt-Winters (HW) and Exponential Trend Smoothing (ETS) models yielded better forecasts than ARIMA for the Thanjavur region, based on criteria like Akaike Information Criteria (AIC) and Root Mean Square Error (RMSE). Lastly, [18] provided an overview of patterns in rainfall, number of rainy days, and temperature variations across India, and stressed the importance of establishing a comprehensive network of baseline stations to support climatic research.

III. OBJECTIVES OF THE STUDY

The basic objectives of this study are mentioned below.

To check out the randomness of data based on the Run test and assumption of identical distributions by using the Mann-Whitney test.

To execute AMRF using FD, WD, LLD and GD based on BE, ME, ML and PWM estimation methods.
To find the most suitable method(s) and distribution(s) using Kolmogorov Smirnov (KS) test for modeling AMRF data.

IV. Materials and Methods

This study deals with the regional rainfall frequency analysis (RRFFA) by using AMRF data from six sites in Punjab namely Lahore, Islamabad, Murree, Jhelum, Mianwali, Sialkot, and two sites of AJ&K i.e., Muzaffarabad and Kotli from the period 1980 - 2015, collected from Pakistan Metrological Department (PMD). The parameters of FD, WD, LLD and GD are computed using ML, BE, ME and PWM methods. The data analysis is presented below.

4.1. Study Area and Data Utilized

Characteristics of this study are presented in Table 1, and observed that Murree has highest elevation of the Sea level that is 2291 meters. Muzaffarabad is considered the second-highest site that is 739m above the sea level. Lahore and Mianwali have got approximately the same elevation (213m and 210m) rather than the sea level. Kotli possessed 609 m elevation than the sea level. Islamabad, Sialkot, and Jhelum have 508m, 256m and 235 meters elevation of the Sea level.

4.1. Descriptive Statistics

The descriptive statistics of AMRF data for Punjab and AJ&K are presented in Table 1. Such as, mean, median, minimum (Min), maximum (Max), Coefficient of Variation (CV), Coefficient of Skewness and Coefficient of Kurtosis is obtained.

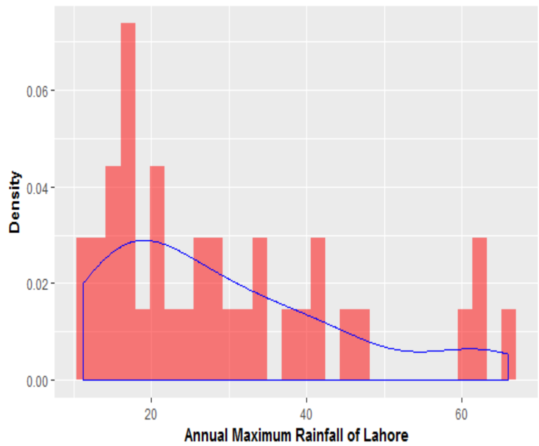
Table 1. Descriptive Statistics for AMRF Data

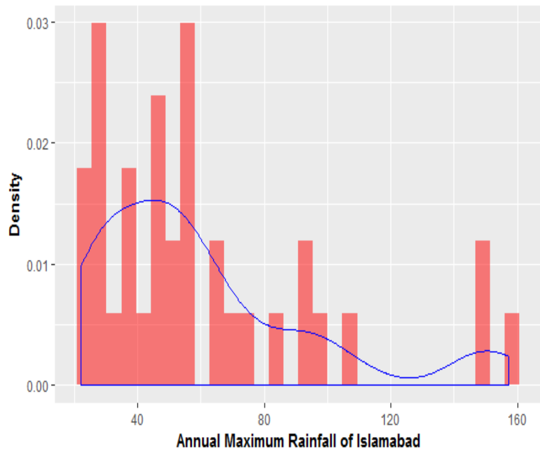
Sites	Mean	Median	Min.	Max.	CV	Skewness	Kurtosis
Lahore	29.1649	25.42085	11.2031	65.9314	53.1738	0.9605	2.9340
Islamabad	60.1880	50.6092	22.1695	157.2260	58.9953	1.3626	4.1983
Murree	75.7221	69.6021	32.1477	184.2700	47.2509	1.0740	3.8669
Jhelum	48.05278	38.9576	21.1314	127.4070	52.8012	1.4383	4.8136
Mianwali	52.8291	47.4246	12.7229	177.2650	62.6911	1.7308	6.9267
Sialkot	43.0479	38.0358	17.1701	114.4230	53.9723	1.5159	5.3007
Muzaffarabad	74.4637	65.7280	30.5490	153.9840	42.2058	1.0452	3.4603
Kotli	59.9135	52.4964	28.7178	129.2760	40.9829	1.2323	3.7635

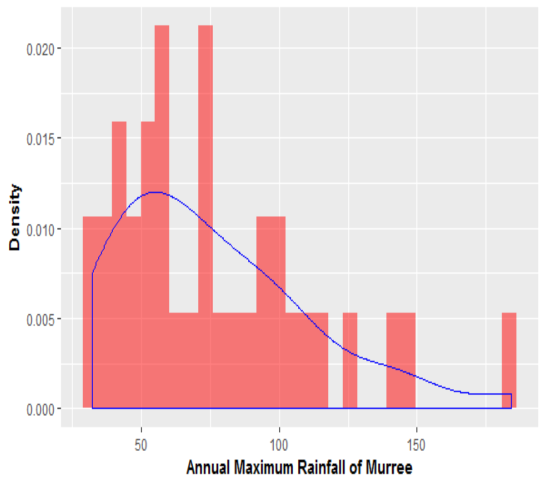
It is shown that the average AMRF ranges from 29.1649mm to 75.7221mm. It is observed that area in which the elevation is higher rather than the sea level has received the maximum rainfall than the other sites. It is also noted that there is an extensive difference of rainfall with CV from 40.9829% to 62.6911%. The rainfall followed a positive extreme with the coefficient of skewness ranging from 1.0030 to 1.8070. This is enough to characterize the distribution as non-normal. This condition is further reinforced by the coefficient of kurtosis which ranges from 2.9340 to 6.9267 indicating that there is maximum peak flow in the data.

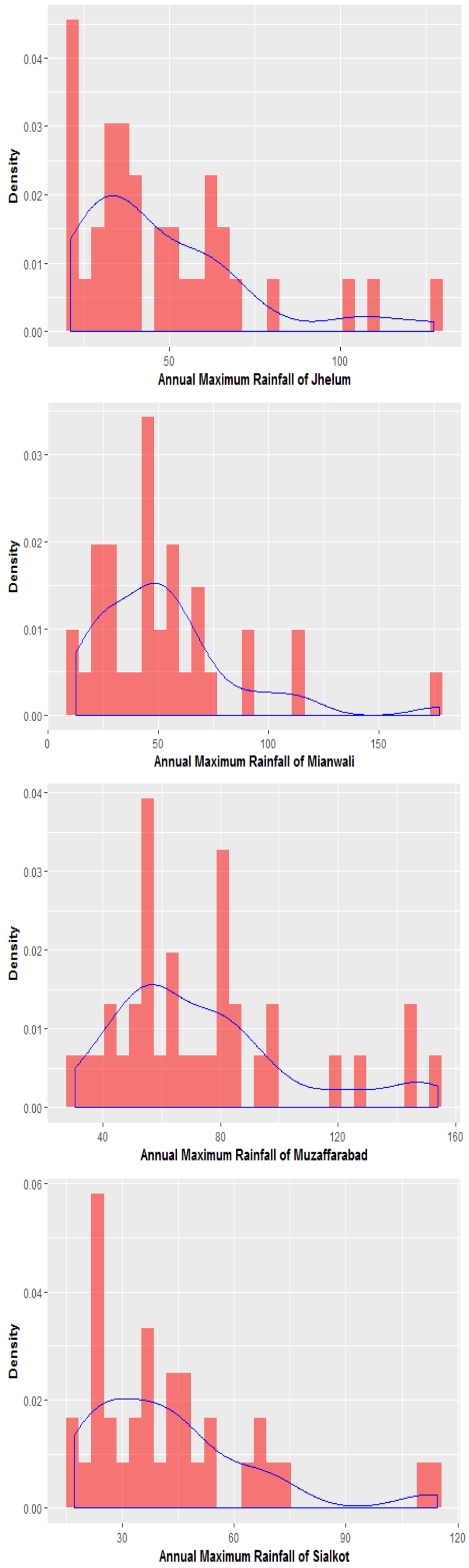
Histogram

Graphical visualization plays an important role in any data analysis. For this purpose, histograms of AMRF along with normal curves were also drawn in Figure 1. Histograms are illustrating the ranges of values of coefficient of skewness and kurtosis. The results showed that distribution of the data for given sites are positively skewed and for kurtosis the distributions of data is leptokurtic.









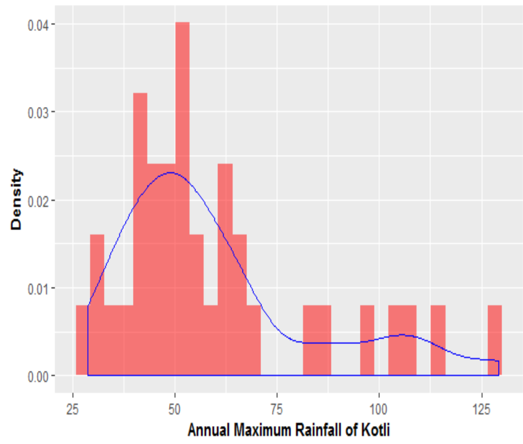


Figure 1. Histograms of AMRF for Lahore, Islamabad, Murree and Jhelum, Mianwali, Muzaffarabad, Sialkot and Kotli.

4.2. Test of Randomness and Identical Distribution

Before fitting the aspirant distribution namely FD, WD, LLD and GD, check the randomness and basic assumptions of identical distribution. Run test is applied to check the randomness of the data and Mann Whitney used for identical of distribution and the results represented in Tables 2 and 3. Table 2, demonstrates the result of randomness and P-values shows that the observed runs of numbers (ORN) to ensure that the acceptance of the hypothesis of randomness at a 5% significance level.

Table 2. Runs Test and P-values of Eight Sites

S#	Sites	ORN	p-values
1	Lahore	17	0.612
2	Islamabad	18	0.866
3	Murree	20	0.866
4	Jhelum	26	0.028
5	Mianwali	8	0.000
6	Sialkot	22	0.398
7	Muzaffarabad	20	0.866
8	Kotli	21	0.612

Table 3 expresses the outcomes of the Mann Whitney (W) statistic. Firstly, we have divided each site of data into two categories. For the acceptance of the hypothesis of identical distribution, the *p*-value provides strong evidence at 5 % level of significance.

Table 3. Mann Whitney (W) Statistics Values with P-values

S. No.	Site Name	Groups	W	P-value
1	Lahore	18,18	138	0.4572
2	Islamabad	18,18	144	0.5690
3	Murree	18,18	156	0.8494
4	Jhelum	18,18	160	0.9495
5	Mianwali	18,18	95	0.034
6	Sialkot	18,18	147	0.635
7	Muzaffarabad	18,18	149	0.680
8	Kotli	18,18	155	0.824

V. CANDIDATE DISTRIBUTIONS AND PARAMETERS ESTIMATION

The candidate distributions are FD, LLD, WD and GD. For the analysis of AMRFA, we have to estimate the parameters of candidate distributions using four estimation methods, that is, ML, ME, BE and PWM in the current study. To illustrate the significance of the estimated parameters p- values of the KS test at 5 % significance level are used. The results of estimated parameters are presented in Tables 5 to 8.

5.1. Frechet Distribution (FD)

In 1927, [19] proposed Frechet distribution to measure extreme events. For example, floods, maximum Rainfall, earthquakes etc. The other name of this distribution is Inverse Weibull distribution and a special case of GEV distribution. Probability density function (PDF) of FD is

$$f(x;\ \alpha,\beta) = \frac{\alpha}{\beta}\left(\frac{\beta}{x}\right)^{\alpha+1} e^{-\left(\frac{\beta}{x}\right)^{\alpha}}, x > 0, \alpha > 0, \beta > 0 \quad (1)$$

Cumulative Distribution Function (CDF) of FD is

$$(x) = exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right), \qquad x > 0 \qquad (2)$$

Where, α and β are the shape and scale parameter of FD. Estimates of FD are presented in Table 4

Table 4. Estimates of Parameters using FD

Sites	Methods	$\hat{\alpha}$	$\hat{\beta}$	KS	P-values
Lahore	ML	2.3329	20.1390	0.1013	0.8539
	BE	2.3000	20.0834	0.1003	0.8617
	PWM	2.6965	20.4530	0.1204	0.6733
Islamabad	ML	2.2208	40.4027	0.0980	0.8467
	BE	2.1900	40.2947	0.0940	0.8790
	PWM	2.5618	41.0146	0.1317	0.5175
Murree	ML	2.5054	55.0043	0.1030	0.8035
	BE	2.4705	54.8954	0.1015	0.8162
	ME	3.7203	60.5173	0.1778	0.1817
	PWM	3.0016	55.9329	0.1282	0.5523
Jhelum	ML	2.5207	34.2504	0.0958	0.8646
	BE	2.4868	34.1820	0.0924	0.8908
	ME	3.4665	37.5400	0.1614	0.2749
	PWM	2.8274	34.5247	0.1192	0.6424
Mianwali	ML	1.7152	33.1231	0.1956	0.1107
	BE	1.6967	32.9068	0.1974	0.1152
	ME	3.1317	39.7295	0.2150	0.0612
	PWM	2.4827	35.3227	0.2212	0.0490
Sialkot	ML	2.4398	30.3455	0.1282	0.5524
	BE	2.4061	30.2686	0.1275	0.5591
	PWM	2.7882	30.7158	0.1617	0.2727
Muzaffara bad	ML	2.7463	56.6662	0.0974	0.8518
	BE	2.7104	56.5793	0.0995	0.8336
	ME	4.0172	60.8366	0.1345	0.4913
	PWM	3.3455	57.4418	0.1269	0.5645
Kotli	ML	3.2387	46.9791	0.0925	0.8819
	BE	3.1941	46.9302	0.0930	0.8868
	ME	4.1011	49.2182	0.1259	0.5749
	PWM	3.5121	47.0088	0.0965	0.8587

Values of estimated parameters using all four methods for all selected sites are presented in Table 4. Also, this table contains test statistics values of KS and P-values. The results indicated that estimated Parameters for FD by using all four selected methods are similar to each other except ME method, which is based on the p -values of the KS test at 5% level of significance and concluded that ML, BE and PWM method are good fitted for FD.

WD is a continuous probability distribution that was proposed by [20]. WD is useful for AMRF data. It is also used in Survival analysis, Reliability engineering, failure analysis etc. The PDF of WD is

$$f(x;\ \alpha,\beta) = \left(\frac{\beta}{\alpha}\right)\left(\frac{\beta}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x}{\alpha}\right)^{\beta}}, x > 0,\ \alpha,\beta > 0 \qquad (3)$$

The corresponding CDF of WD is

$$F(x) = 1 - e^{-\left(\frac{x}{\alpha}\right)^{\beta}}, \qquad (4)$$

Where, α is the shape and β is the scale parameters of WD. Estimates of WD are mentioned in Table 5.

5.2. Weibull Distribution (WD)

Table 5. Estimates of Parameters for WD

Sites	Sites	Methods	$\hat{\alpha}$	$\hat{\beta}$	KS
Lahore	ML	2.0548	33.1267	0.1185	0.6931
	BE	2.0136	32.7500	0.1089	0.7865
	ME	1.9933	32.9070	0.1102	0.7746
	PWM	1.9982	32.9085	0.1096	0.7798
Islamabad	ML	1.8658	68.3436	0.1469	0.3820
	BE	1.8280	67.3519	0.1356	0.4829
	ME	1.7767	67.6367	0.1355	0.4819
	PWM	1.8628	67.7811	0.1417	0.4261
Murree	ML	2.2880	85.7863	0.1089	0.7489
	BE	2.2467	84.9466	0.1066	0.7888
	ME	2.2735	85.4837	0.1052	0.7821
	PWM	2.3044	85.4715	0.1338	0.7534
Jhelum	ML	2.0592	54.5717	0.1321	0.5136
	BE	2.0222	53.9763	0.1394	0.4462
	ME	2.0090	54.2261	0.1398	0.4425
	PWM	2.1296	54.2583	0.1338	0.4977
Mianwali	ML	1.7517	59.7160	0.1175	0.6598
	BE	1.7190	58.8426	0.1108	0.7273
	ME	1.6618	59.1118	0.1172	0.6624
	PWM	1.7833	59.3784	0.1122	0.7137
Sialkot	ML	2.0139	48.8773	0.1214	0.6202
	BE	1.9814	48.2960	0.1232	0.6020
	ME	1.9606	48.5543	0.1243	0.5912
	PWM	2.0902	48.6020	0.1184	0.6511
Muzaffar abad	ML	2.5398	84.0473	0.1245	0.5887
	BE	2.4937	83.4188	0.1175	0.6596
	ME	2.5790	83.8555	0.1215	0.6192
	PWM	2.6492	83.7871	0.1200	0.6339
Kotli	ML	2.5927	67.6188	0.1630	0.2641
	BE	2.5492	67.1179	0.1541	0.3254
	ME	2.6651	67.4021	0.1652	0.2504
	PWM	2.8161	67.2697	0.1731	0.2053

Estimated results of shape (α) and scale (β) parameters for WD are presented in Table 5. P-values of KS test depict that WD is well fitted for all the selected sites. Hence, from the all-selected methods BE is the best with highest p-value for the fitting of WD on AMRF data.

5.3. Log-logistic Distribution (LLD)

LLD introduced by [21]. This is continuous probability distribution for non-negative random variables. The LLD is used for modeling stream flow rates and precipitation, for example, AMRF, maximum daily rainfall, maximum monthly rainfall, river flow per month or per year etc. The PDF of LLD is,

$$f(x;\alpha,\beta) = \frac{\left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\beta-1}}{\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{2}}, \quad x > 0,\ \alpha,\beta > 0 \qquad (5)$$

The corresponding CDF of LLD is

$$F(x) = \frac{1}{1+\left(\frac{x}{\alpha}\right)^{-\beta}}, \qquad (6)$$

Where, α and β is the scale and shape parameters for LLD.

Estimates of LLD are given below in Table 6.

Table 6. Estimates of the Parameters for LLD

Sites	Sites	Methods	$\hat{\alpha}$	$\hat{\beta}$	KS
Lahore	ML	25.2009	3.3302	0.1045	0.8271
	BE	25.0095	3.2695	0.0959	0.8950
	ME	26.1846	3.9491	0.1652	0.2794
	PWM	25.2142	3.4116	0.1097	0.7790
Islamabad	ML	50.9335	3.2582	0.0865	0.9294
	BE	50.5179	3.1988	0.0787	0.9663
	ME	53.0636	3.6613	0.1291	0.5430
	PWM	51.0747	3.2184	0.0849	0.9379
Murree	ML	67.9686	3.7961	0.0795	0.9631
	BE	67.5740	3.7258	0.0724	0.9847
	ME	69.2261	4.3226	0.1226	0.7129
	PWM	67.5926	3.8496	0.0759	0.9757
Jhelum	ML	41.9301	3.6729	0.0889	0.9142
	BE	41.6769	3.6051	0.0866	0.9286
	ME	43.1843	3.9679	0.1223	0.6113
	PWM	42.1800	3.5994	0.0930	0.8862
Mianwali	ML	45.2447	2.9867	0.1390	0.4501
	BE	44.7720	2.9280	0.1466	0.3838
	ME	46.0600	3.5120	0.1234	0.5999
	PWM	44.2660	3.1051	0.1559	0.3123
Sialkot	ML	37.5904	3.5836	0.1246	0.5880
	BE	37.3292	3.5157	0.1171	0.6643
	ME	38.5535	3.9056	0.1545	0.3223
	PWM	37.6251	3.5432	0.1225	0.6091
Muzaffarabad	ML	68.0016	4.3984	0.0855	0.9346
	BE	67.7250	4.3139	0.0788	0.9659
	ME	69.1210	4.7358	0.1145	0.6902
	PWM	68.1397	4.3438	0.0852	0.9363
Kotli	ML	54.3723	4.8122	0.0906	0.9035
	BE	54.1898	4.7197	0.0870	0.9262
	ME	55.8203	4.8558	0.1223	0.6113
	PWM	55.3309	4.5833	0.1116	0.7199

Estimates of LLD (shape and scale parameters) using four methods of estimation are given in Table 6 and seen that the results of the shape and scale parameters are not significantly different on the basis of KS test and their corresponding P-values. It is also indicated that BE method is the best among all four methods for LLD.

5.4. Gamma Distribution (GD)

GD introduced by [22], it is a continuous probability distribution frequently used for extreme events, such as time to failure of equipment, load levels of telecommunication and maximum rainfall etc. The PDF of GD is,

$$f(x;\ \alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma\alpha} x^{\alpha-1} e^{-\beta\, x},\ x > 0, \alpha, \beta > 0 \qquad (7)$$

The CDF of GD is,

$$F(x) = \frac{\gamma(\alpha,\beta x)}{\Gamma\alpha}, \quad \text{Where,} \qquad (8)$$

$$\gamma(\alpha,\beta x) = \int_0^x \beta^{\alpha} x^{\alpha-1}\, e^{-\beta x} dx,$$

Estimates of GD are given below in Table 7.

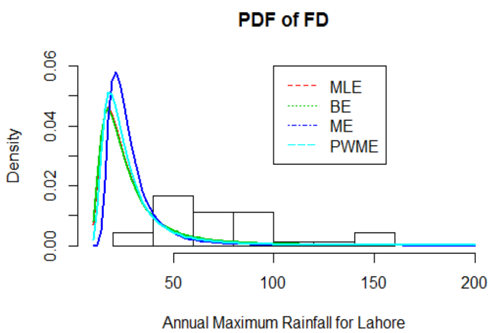
Table 7. Estimates of Parameters of GD

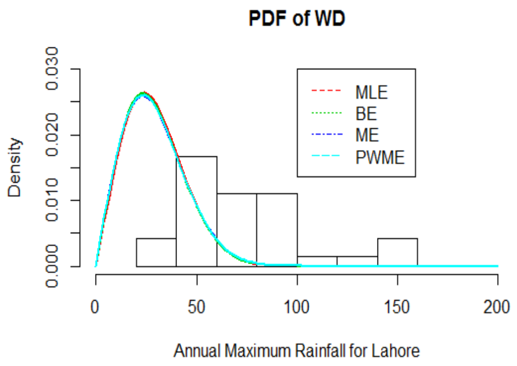
Sites	Sites	Methods	$\hat{\alpha}$	$\hat{\beta}$	KS
Lahore	ML	4.0741	7.1582	0.1193	0.6848
	BE	3.8629	7.6046	0.1159	0.7192
	ME	3.6378	8.0193	0.1028	0.8414
	PWM	3.4464	8.4624	0.0952	0.9001
Islamabad	ML	3.6073	16.6945	0.1259	0.5747
	BE	3.4270	17.6991	0.1289	0.5453
	ME	2.9553	20.3666	0.1157	0.6684
	PWM	3.0376	19.8143	0.1171	0.6738
Murree	ML	5.1311	14.7493	0.1039	0.7938
	BE	4.8594	15.6740	0.1019	0.8123
	ME	4.6070	16.4474	0.0911	0.8999
	PWM	4.4606	16.9758	0.0873	0.9246
Jhelum	ML	4.5136	10.6496	0.1294	0.5402
	BE	4.2825	11.2867	0.1279	0.5549
	ME	3.6893	13.0208	0.1142	0.6931
	PWM	3.8665	12.4281	0.1123	0.7125
Mianwali	ML	3.1086	17.0068	0.1095	0.7400
	BE	2.9498	18.0832	0.1081	0.6466
	ME	2.6171	20.2020	0.1252	0.5815
	PWM	2.8089	18.8080	0.1188	0.7539
Sialkot	ML	4.3361	9.9305	0.1167	0.6681
	BE	4.1113	10.5374	0.1106	0.7290
	ME	3.5310	12.1951	0.1034	0.7990
	PWM	3.7383	11.5153	0.0940	0.8789
Muzaffarabad	ML	6.4574	11.5340	0.0960	0.8632
	BE	6.1100	12.2399	0.0916	0.8962
	ME	5.7742	12.9032	0.0889	0.9148
	PWM	5.7509	12.9483	0.0890	0.9158
Kotli	ML	7.2496	8.2645	0.1460	0.3894
	BE	6.8654	8.7566	0.1457	0.3912
	ME	6.1240	9.7847	0.1347	0.4891
	PWM	6.4320	9.3150	0.1380	0.4590

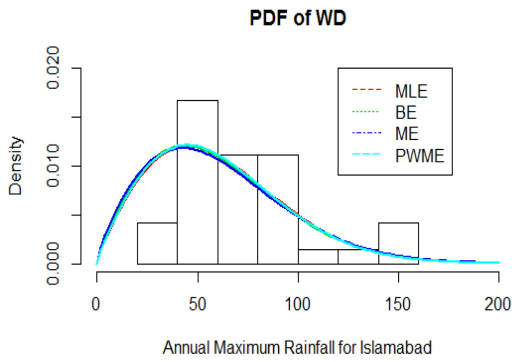
Table 7 illustrated the results of GD by using the four methods of estimation. P-values of KS test shows that GD is good fit for AMRF data of all sites at 5 percent significance level. P-values of KS test indicate that estimation through PWM method is best for GD of AMRF data for all sites

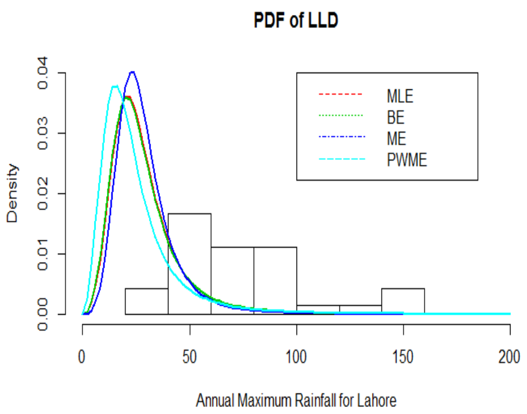
VI. GRAPHS OF PROBABILITY DENSITY FUNCTION (PDF)

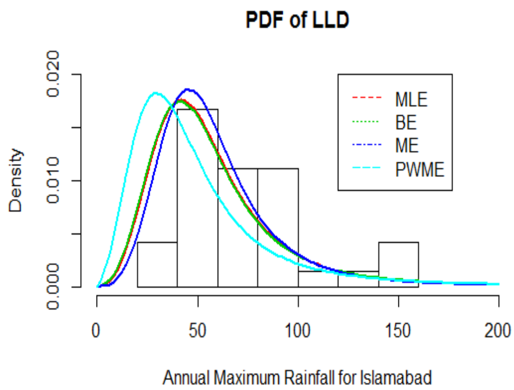
The graphical performances of the mentioned probability distributions with four methods of estimations have been presented in Figures 2 to 5. These graphs suggested that, in general; the given distributions are suitable for observed data series at given sites. Particularly, for WD, the behavior of ME method is relatively different for the site Muzaffarabad whereas for other sites the performance of MLE, BE and PWME methods are approximately similar. The results are reliable with the conclusions of KS test.

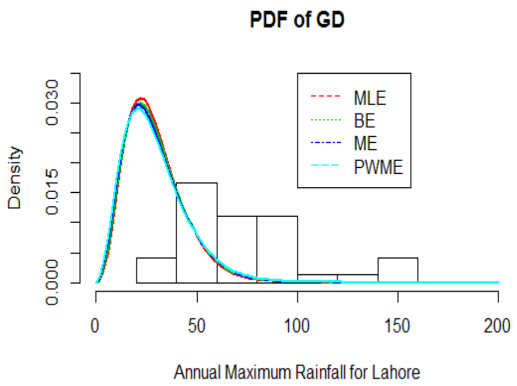


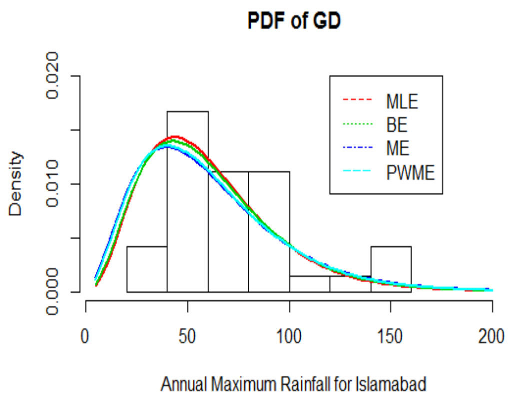

PDF of WD
MLE
BE
ME
PWME
Density
Annual Maximum Rainfall for Lahore

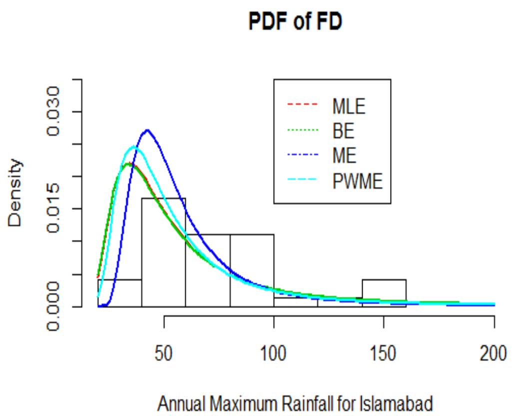

PDF of WD
MLE
BE
ME
PWME
Density
Annual Maximum Rainfall for Islamabad

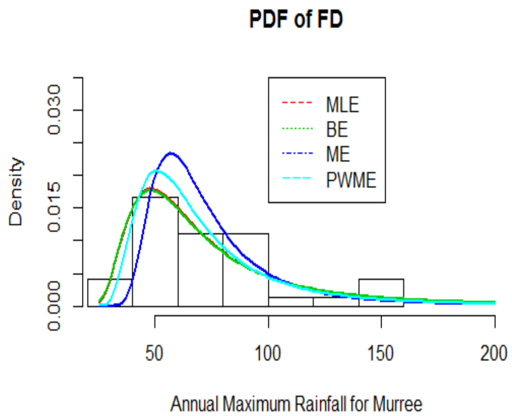

PDF of LLD
MLE
BE
ME
PWME
Density
Annual Maximum Rainfall for Lahore

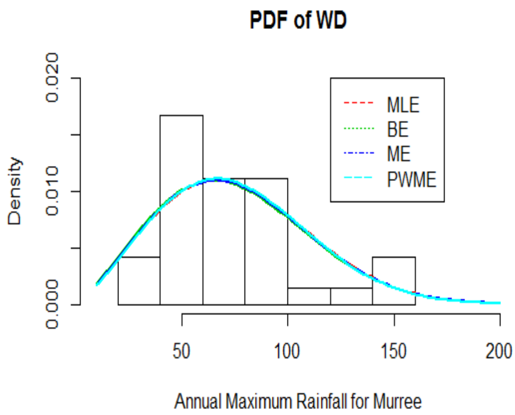

PDF of LLD
MLE
BE
ME
PWME
Density
Annual Maximum Rainfall for Islamabad

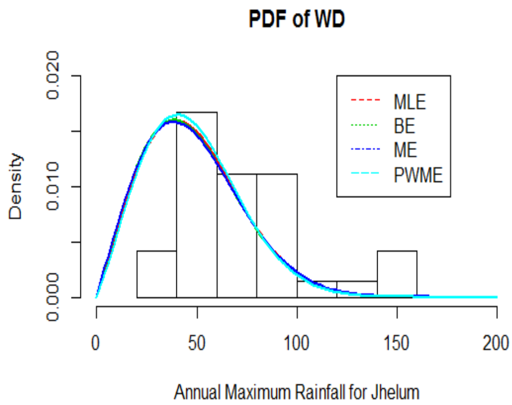

PDF of GD
MLE
BE
ME
PWME
Density
Annual Maximum Rainfall for Lahore

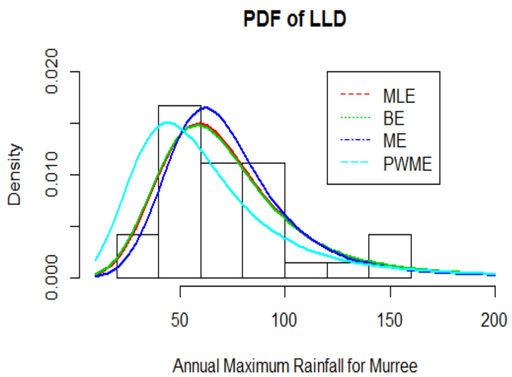

PDF of GD
MLE
BE
ME
PWME
Density
Annual Maximum Rainfall for Islamabad

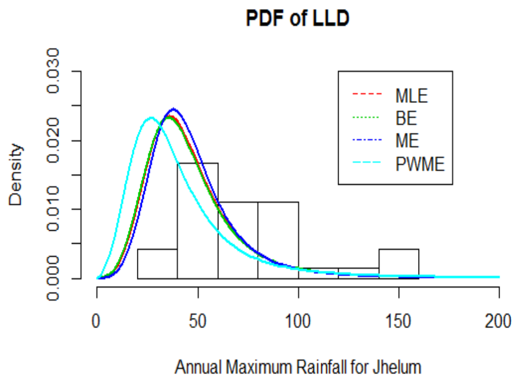

PDF of FD
MLE
BE
ME
PWME
Density
Annual Maximum Rainfall for Islamabad

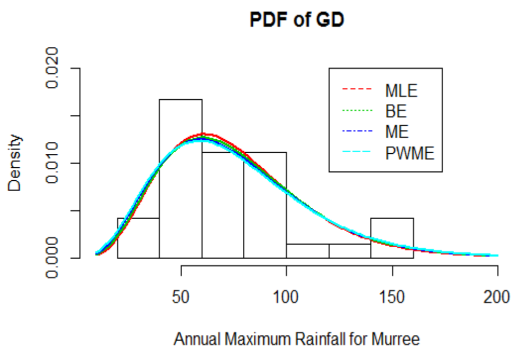

PDF of FD
MLE
BE
ME
PWME
Density
Annual Maximum Rainfall for Murree











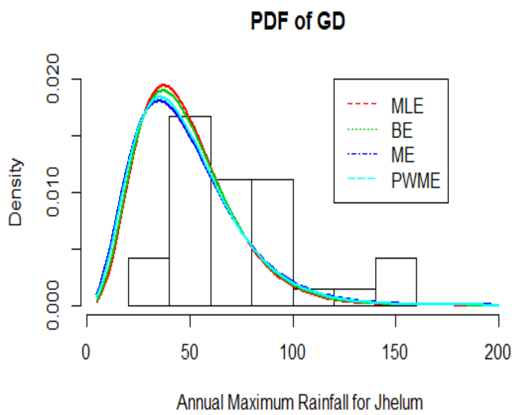
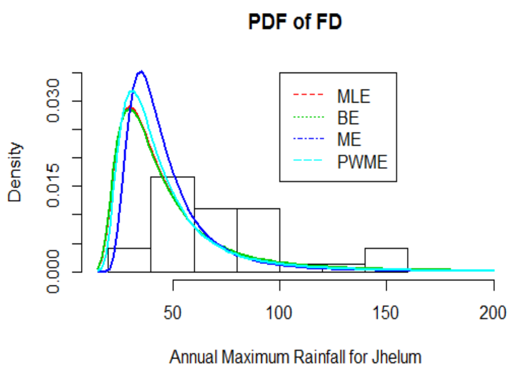
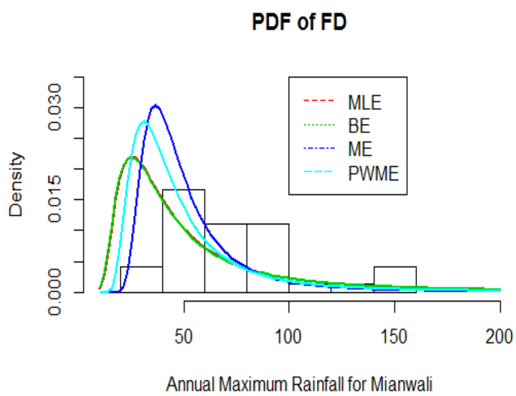
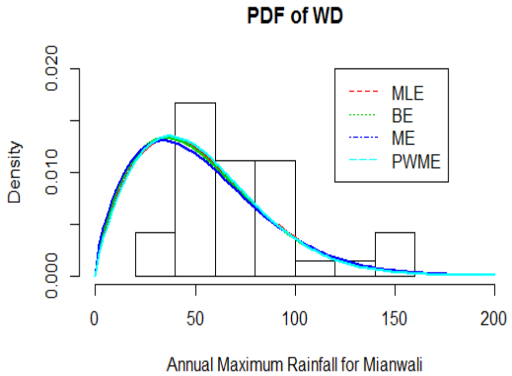
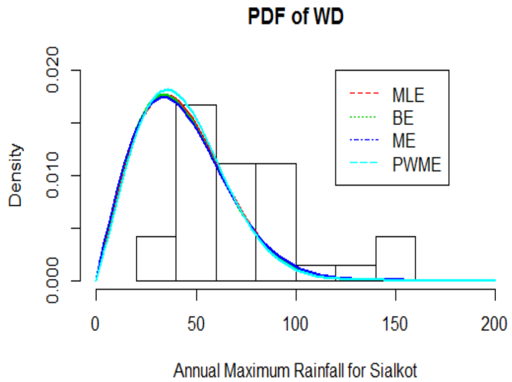


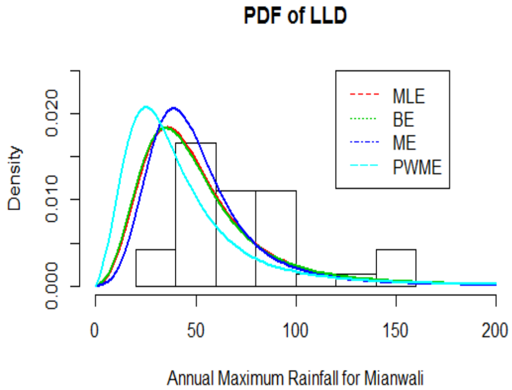
Figure. 2 PDF Graphs for Lahore, Islambad and Muree

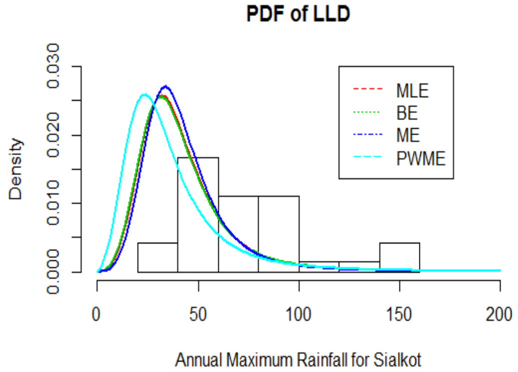


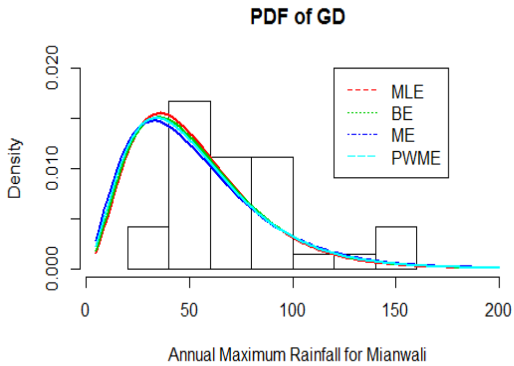












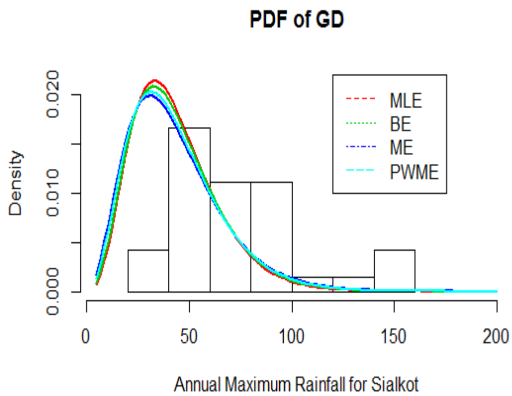
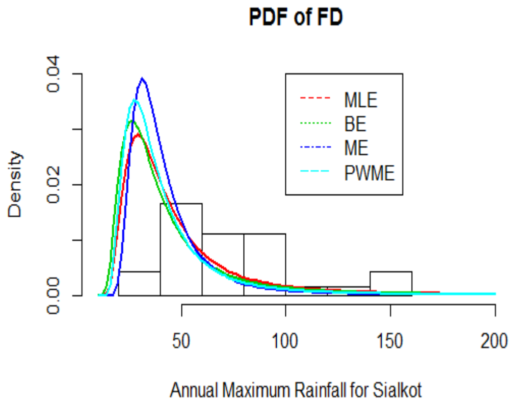
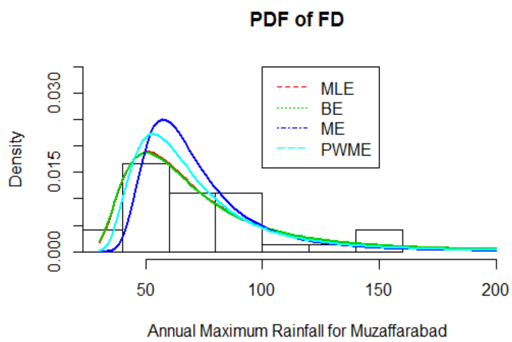
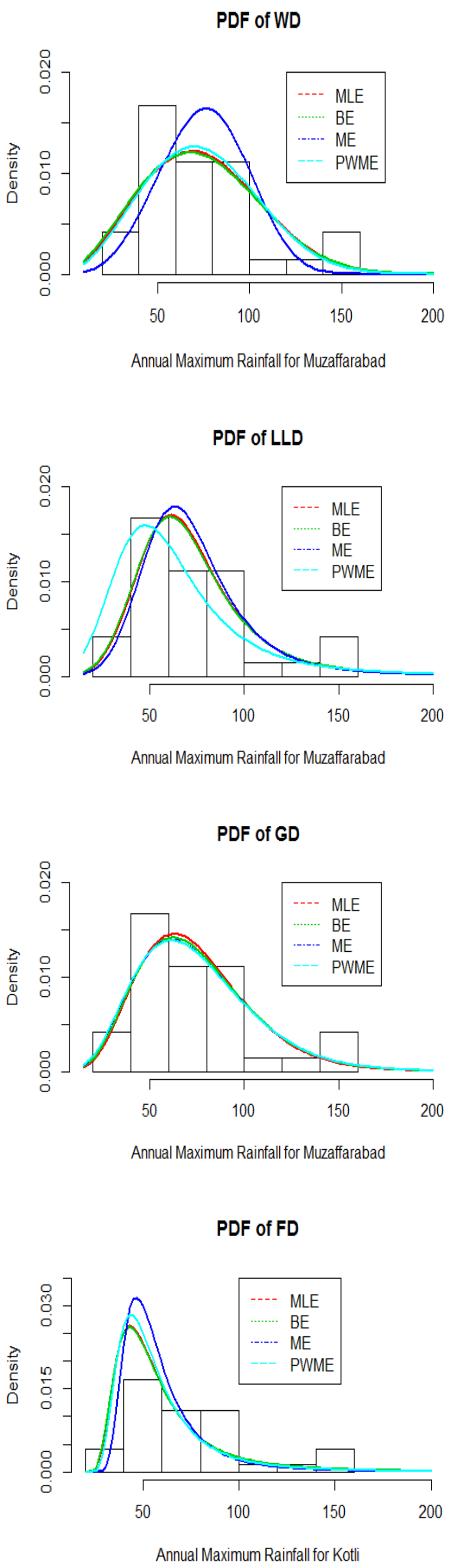


Figure 3. PDF graph for Jhelum, Mainwali and Sialkot







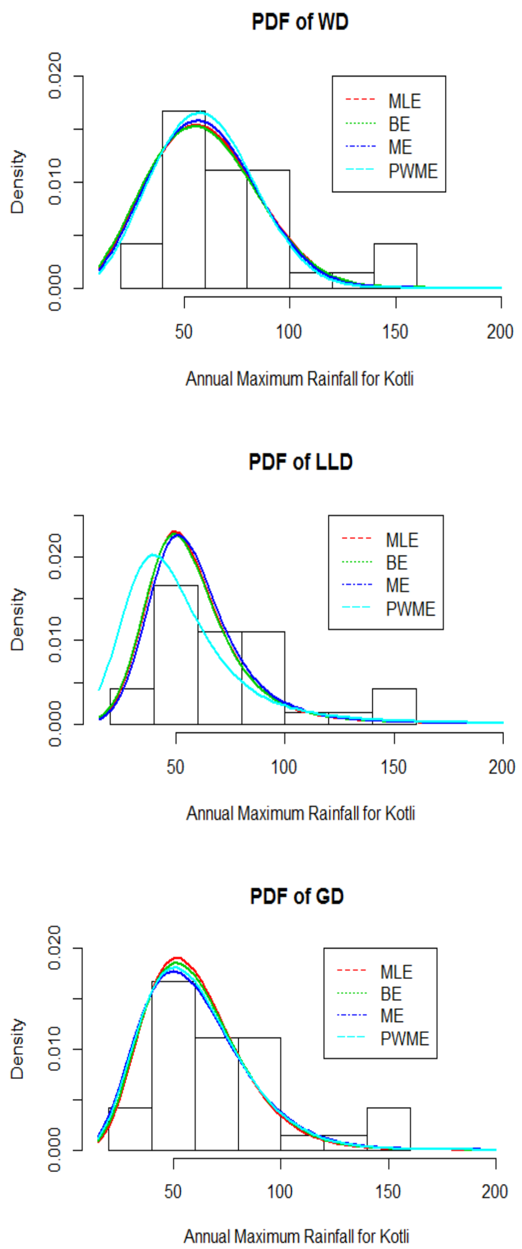
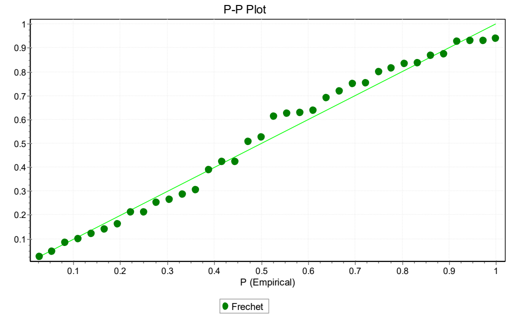
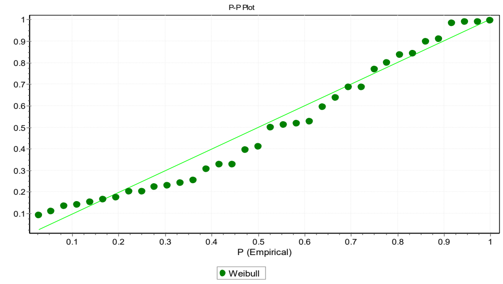


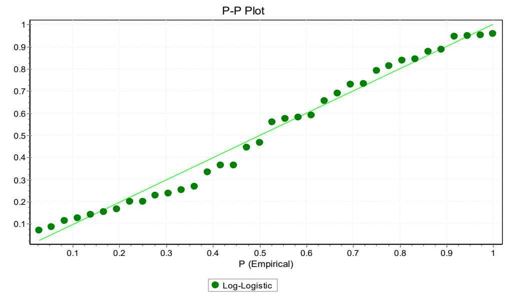
Figure 4. PDF for Muzaffarabad and Kotli

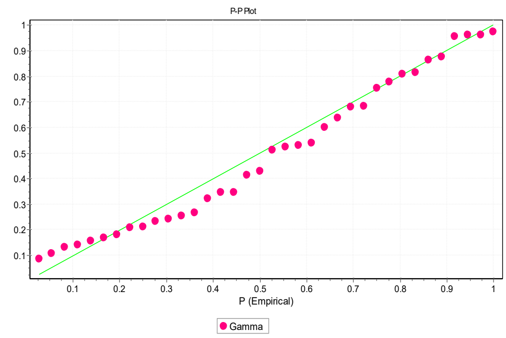
6.1. Probability–Probability (PP) Plots

Figure 5 presents the PP plots for Annual Maximum Rainfall (AMRF) across eight different cities. These plots are used to assess how well the observed rainfall data corresponds with the values predicted by the chosen probability distribution. In cities such as Murree, Muzaffarabad, and Sialkot, the data points closely follow the diagonal line, suggesting that the fitted distribution effectively represents the actual rainfall behavior in those locations. Overall, these PP plots serve as a visual tool to evaluate the accuracy of the selected statistical model in capturing rainfall extremes, thereby enhancing confidence in subsequent hydrological assessments and risk evaluations.

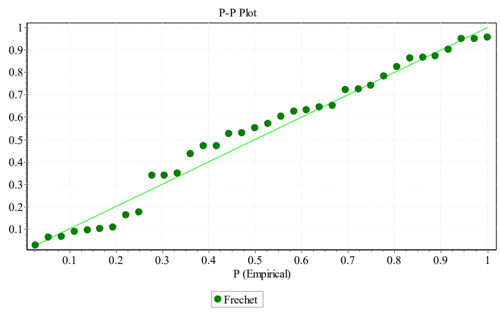


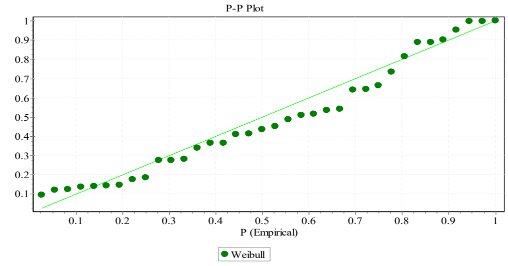


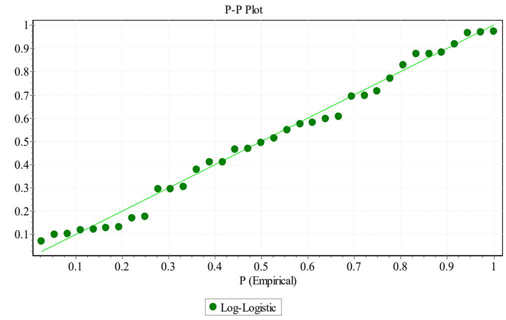


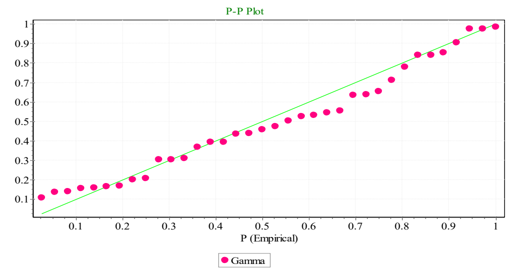


Lahore

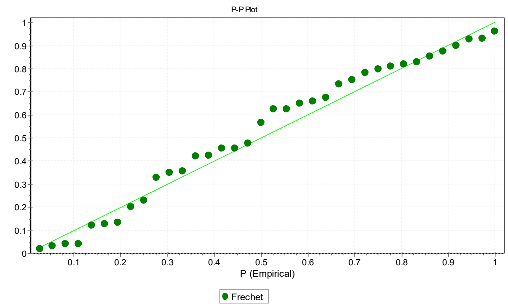


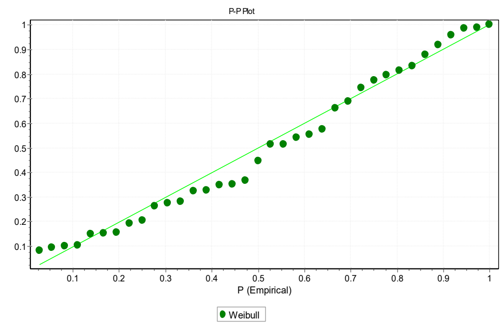


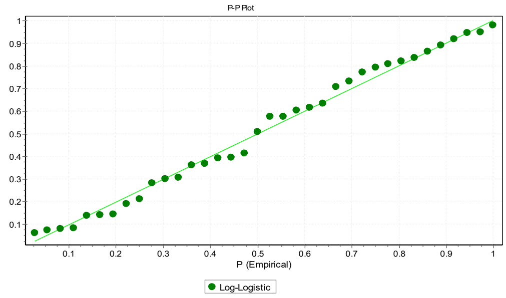


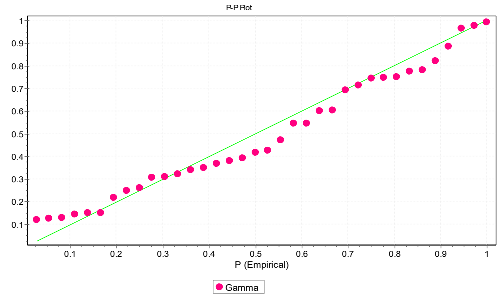


Islamabad

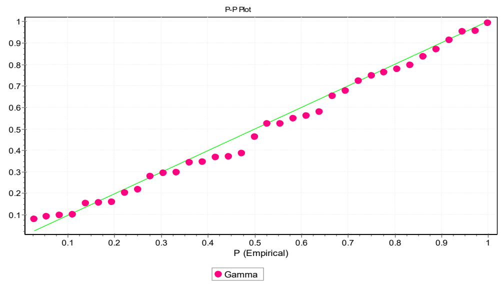




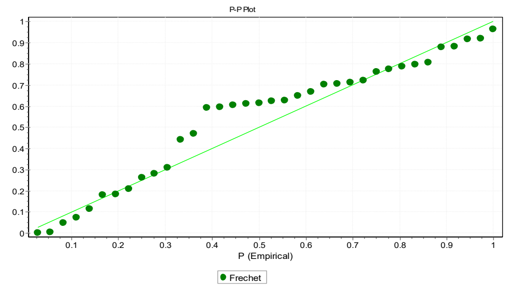


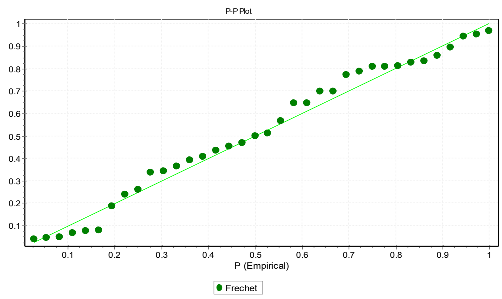


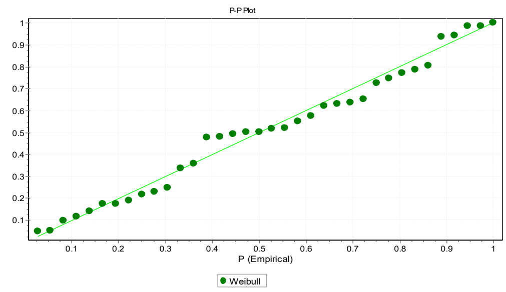
Jhelum

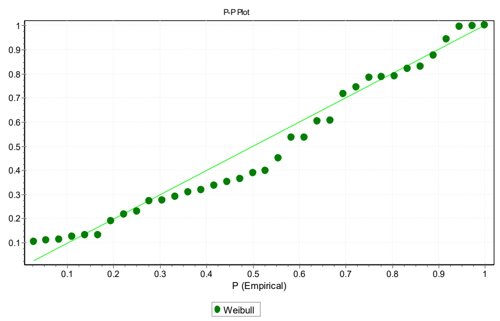


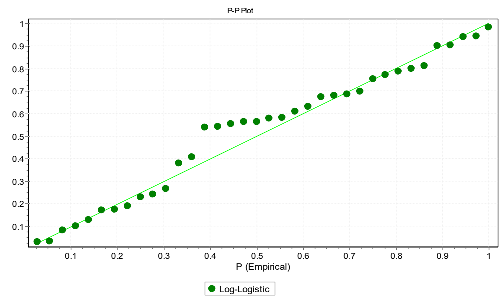
Murree

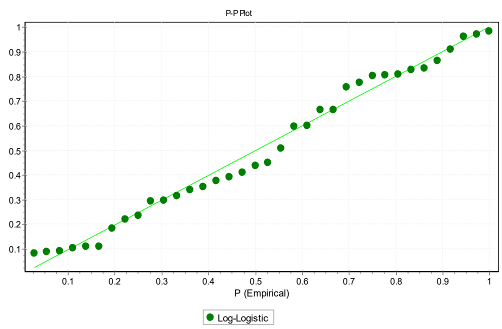


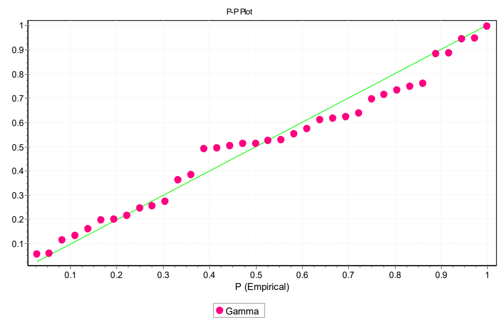




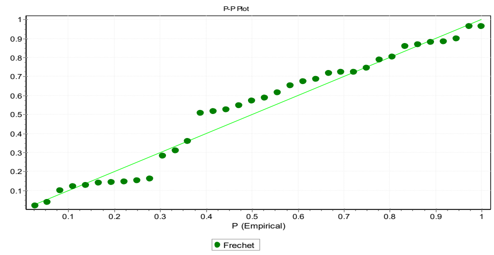


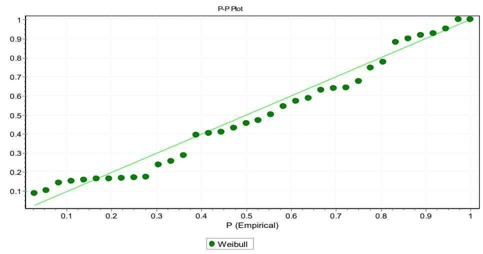


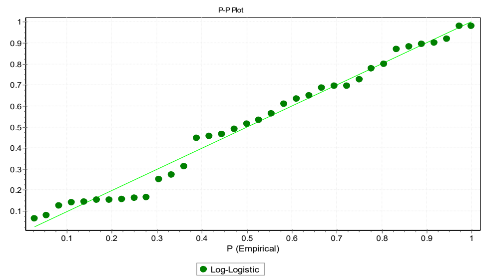


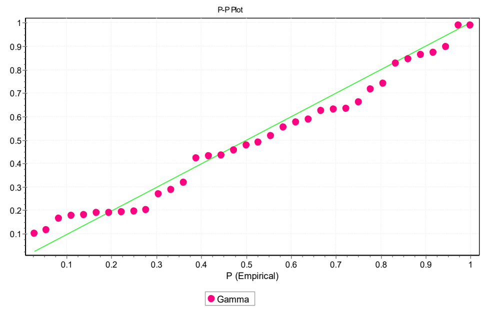


Mianwali

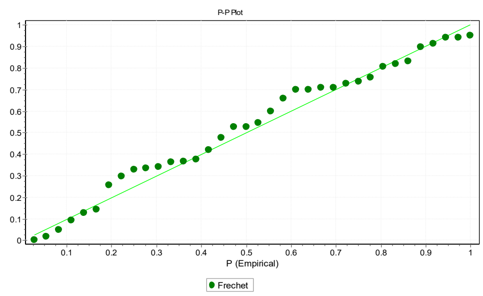


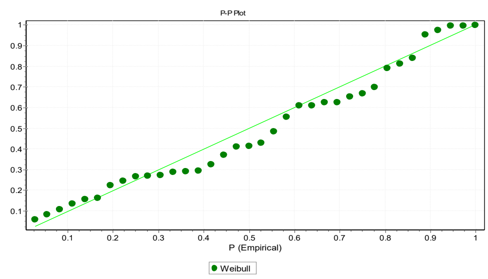


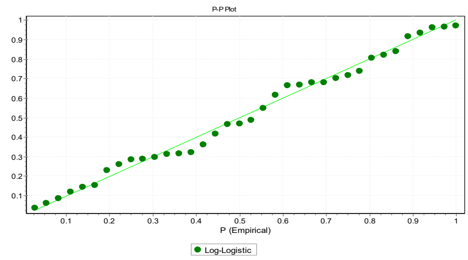


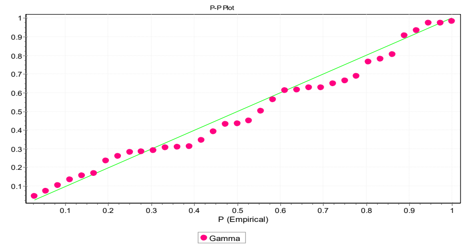


Sialkot

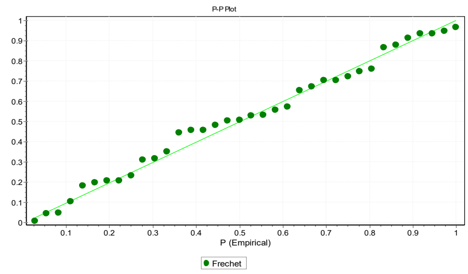


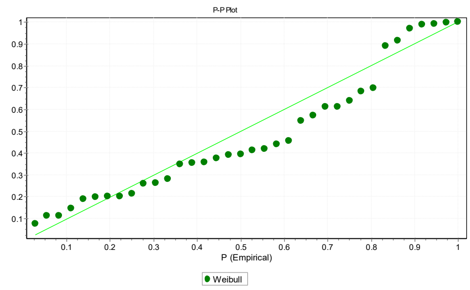


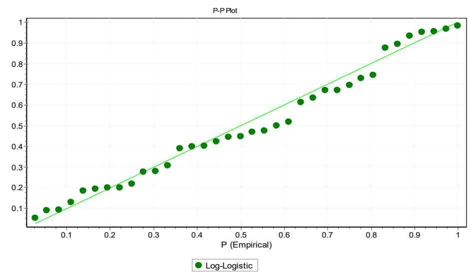


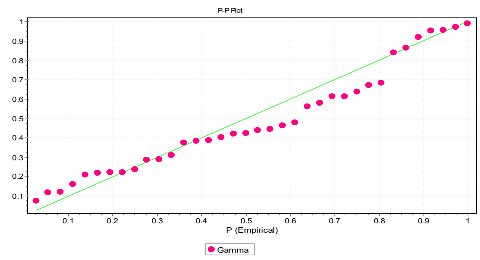


Muzaffarabad









Kotli

Figure 5. PP Plots of AMRF for Lahore, Islamabad, Murree, Jhelum, Mianwali, Muzaffarabad, Sialkot and Kotli.

The performance of all four selected distributions using four methods of estimations is presented in Tables 4 to 7. Moreover, the PDF plots of the candidate distributions are presented in Figures 2 to 5. It can be seen that the candidate distributions are appropriate for the observed data of all eight sites. These results are approximately the same as the findings obtained by the KS test.

Hence, from the above-mentioned details, it is concluded that to model the AMRF data all methods of estimations for WD, LLD, and GD are found good fitted for each site but for FD, the method of ME is not appropriate for Lahore, Islamabad and Sialkot sites similarly method of PWM is not suitable for Mianwali site.

VII. QUANTILES

Using quantile functions for these candidate probability distributions, firstly, estimate the quantiles for several time periods. These results of quantiles have been considered for the most suitable value of BE, MLE, ME and PWME methods. Firstly, all four methods were compared with each other for every site one by one on the bases of p-values then quantiles were estimated only for that method which has the highest p-value. The final results are given in Tables 8 to 11. It is observed that the estimated quantiles for F = 0.9 (return period = 10 years) and the upcoming values are greater than the average value of the AMRF series at particular sites. Also, the return period equal to 100 years, the magnitude is larger than the maximum value of the given AMRF series (see Table 8-11) for all sites (for all four distributions).

Table 8. Quantiles of FD

Quantile Estimates for FD (mm) with Corresponding (Years)					
Sites	**Method**	**0.1(1)**	**0.5(2)**	**0.9(10)**	**0.99(100)**
Lahore	BE	10.34	23.55	53.42	148.41
Islamabad	BE	20.06	47.63	112.59	329.22
Murree	BE	29.59	63.68	136.50	353.35
Jhelum	BE	18.50	39.61	84.49	217.35
Mianwali	BE	13.38	40.84	123.97	495.18
Sialkot	BE	16.05	35.25	77.13	204.79
Muzaffarabad	BE	32.21	64.77	129.79	308.85
Kotli	BE	29.09	52.64	94.94	198.12

Table 9. Quantiles of WD

Quantile Estimates for FD (mm) with Corresponding (Years)					
Sites	**Method**	**0.1(1)**	**0.5(2)**	**0.9(10)**	**0.99(100)**
Lahore	BE	3.33	27.71	49.71	69.65
Islamabad	BE	5.43	55.11	106.29	155.30
Murree	BE	10.96	72.16	123.13	167.63
Jhelum	BE	5.54	45.02	81.53	114.86
Mianwali	BE	4.05	47.54	95.58	143.06
Sialkot	BE	4.73	40.14	73.57	104.38
Muzaffarabad	BE	13.18	72.01	116.55	153.89
Kotli	BE	11.04	58.12	93.09	122.18

Table 10. Quantiles of LLD

Quantile estimates for LLD(mm) with corresponding (Years)					
Sites	**Method**	**0.1(1)**	**0.5(2)**	**0.9(10)**	**0.99(100)**
Lahore	BE	6.13	25.01	48.97	101.97
Islamabad	BE	12.01	50.52	100.41	212.48
Murree	BE	19.69	67.57	121.87	231.96
Jhelum	BE	11.65	41.68	76.66	149.09
Mianwali	BE	9.32	44.77	94.82	215.07
Sialkot	BE	10.10	37.32	69.73	137.92
Muzaffarabad	BE	23.34	67.72	112.70	196.49
Kotli	BE	20.47	54.19	86.32	143.47

Table 11. Quantile of GD

Quantile estimates for GD(mm) with corresponding (years)					
Sites	**Method**	**0.1(1)**	**0.5(2)**	**0.9(10)**	**0.99(100)**
Lahore	PWM	5.41	26.54	49.68	75.88
Islamabad	PWM	8.89	53.72	106.49	167.81
Murree	PWM	17.41	70.14	123.75	182.85
Jhelum	PWM	9.533	43.98	80.813	122.18
Mianwali	PWM	6.99	46.71	95.08	151.97
Sialkot	PWM	8.22	39.27	72.89	110.83
Muzaffarabad	PWM	21.43	70.19	115.99	164.93
Kotli	PWM	18.78	56.83	91.47	128.03

The quantile estimates for each site with return periods 1, 2, 10 and 100 years for the candidate distributions are obtained and shown in Tables (8 – 11). Overall, the results obtained by FD, WD, LLD, and GD are closed to the observed values.

Since, the above calculated results are suitable for the executives and investigators, dealing with water resources management (WRM) and disaster of the study for planning and effective supervision to make appropriate strategies for the future.

VIII. CONCLUSIONS

This study illustrates the behaviour of all four probability distributions with mentioned estimation methods for AMRF in eight sites of Punjab and AJ&K.

i. Summary statistics and histograms have shown that the variations occur in the selected data series and the distribution related to the observed data series mentioned that the selected locations are positively skewed.
ii. Run test and Mann Whitney tests are used for randomness and for the assumption of identical distributions respectively. p-value for Run test showed that the data of AMRF follow the random pattern. The Mann-Whitney test indicates that given sites are identical.
iii. The candidate distributions with given methods of the estimation of parameters have accepted the GOF criteria by using KS test and the p-values have suggested that three distributions WD, LLD and GD with all four methods of estimation are found good fit but the results showed the estimated parameter for FD using the mentioned methods are relatively similar

except ME method on the basis of P-values of KS test at 5% significance level. It is concluded that ML, Bayesian, PWM are best suited for FD than ME.

iv. The Quantiles have been calculated by using PWME and BE methods and observed that the estimated quantiles for F=0.9, for 10 years return period and for the return period is equal to 100 years, the magnitude is greater than the maximum value of the calculated AMR series for all sites using all four distributions.

Practical Application
This study provides suitable plans for the concerned representatives for the study area, domestic farmers, meteorologists and rainwater management planning. Also, this study provides suitable illustration of some new probability distributions with different methods for parameter estimation, at-site frequency analysis of extreme events, like earthquake, floods, rainfall etc.

Limitations of this Study
This study was only for four distributions. The numbers of distributions that can be used to study the AMRF to extend this study.

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