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On Nearly c-permutable Subgroups of Finite Groups

A.Mahboob[1], I.Ali[2], K.Abbas[3], T.Hussain[4], F.Shahzad[5], M.Altaf[6]

[1] University of Education Lahore, Vehari Campus, Pakistan,
[2] Department of Mathematics and Statistics, The University of Lahore, Pakistan,
[3] Chairmain,Department of Statistics, AJK University Muzafarabad, Kashmir,
[4] University of Education Lahore, Okara Campus, Pakistan,
[5] Beijing University of Chemical Technology, China,
[6] University of Engineering and Technology Taxila, Pakistan.
[1] abid.mahboob@ue.edu.pk,

Abstract- Let H, Q are the subgroups of a finite group G. If HQ $\trianglelefteq$ G and H ∩ Q ≤ H_{ncG} where H_{ncG} is generated by those subgroups of H which are weakly c-permutable in G, then H is said to be nearly c-permutable in G. We give some characterizations of a group by using nearly c-permutablity of groups.
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I. Introduction

All the groups of this article are con-sider to be finite. cc-permutable, is dis- cussed in [1]. Here we will denote sylow-p subgroups by H syl_p ≤ G, normal sylow p-subgroups by H $sly_p \trianglelefteq$ G, Hall sub-group by H ≤ Hall G, normal Hall sub- group by H $\trianglelefteq$ Hall G, FiGing subgroup by Fit(H), generalized fiGing subgroups by Fit∗(H), prime numbers are represented by p, q. This new idea is an ex-tension of [1] and is used to prove some interesting results on the structure of finite permutable groups. Ballester Bol, Y. Wang [2] give the idea of c-normal and c-suplemented subgroups. The idea of weakly c-permutable is introduced by W. Guo. Aim of this paper is to expand the above concepts that can be helpful in further studies and it will vast the new dimensions of permutability. Now we present the following new idea which is an extended form of weakly c-permutable subgroups.

Deftnition 1.1. Let H, Q are the sub-groups of a finite group G. If HQ $\trianglelefteq$ G and H ∩ Q ≤ H_{ncG} where H_{ncG} is generated by those subgroups of H which are weakly c-permutable in G, then H is said to be nearly c-permutable in G. Every weakly c-permutable subgroup is always nearly c-permutable subgroup. But converse does not hold.
Example 1.2. $A_4 \trianglelefteq S_4$ and $A_4 = C_3\ V_4$, $C_3 \cap V_4 = 1 \leq H_{ncG}$. C_3 is nearly c-permutable in S_4. Since $C_3\ V_4 \trianglelefteq G \neq G$. So C_3 is not weakly c-permutable.
Theorem 1.3. Let q divides | G | in such a way (q^2 - 1, |G|. Then there will be X $\trianglelefteq$ G with G/X is q-nilpotent and every subgroup of X of order q^2 is nearly c-permutable in G if and only if G is q-nilpotent. We prove these theorems and then we generalize some results. We use notations of [8].

II. Preliminaries

Lemma 2.1. Suppose that G be a finite group.
(1) If L ≤ Y ≤ G and L nearly c-permut able G implies L nearly c-permutable Y.
(2) If L ≤ Y and L ⊴ G. Implies L nearly c-permutable G iff Y/L nearly c-permutable G/L.
(3) If L ⊴ G and Y be a nearly c-permutable subgroup in G satisfying | | | | (|L| , |Y|) = 1, then the subgroup LY/L nearly c-permutable G/L.

Proof:

(1) If L ∩ V is cc-permutable in G and LV ⊴ G, then

$$Y = Y \cap G = Y \cap LV = L(Y \cap V)$$

and L ∩ (Y ∩ V) = L ∩ V is cc-permutable in Y by ([5,6]). So L nearly c-permutable Y.

(2) Let Y/L is nearly c-permutable within G/L and let G/L has a subnormal sub-group of V/L such that (Y/L)(V/L) = G/L and (V/L) ∩ (Y/L) = (V ∩ Y)/L is cc-permutable in G/L. By ([5,6]) LV ⊴ G and L ∩ V is cc-permutable.
Conversely we assume L nearly c-permutable G, so we can take V ≤ G in such a way LV ⊴ G and L ∩ V is cc-permutable in G. Implies (V/L)(Y/L) ⊴ G and V/L ∩ YL/L = L(Y ∩ V)/L. By Lemma ([5,6]), L(Y ∩ V)/L is cc-permutable in G/L. So Y/L nearly c-permutable G/L.

(3) Suppose L nearly c-permutable G and assume V ⊴ G in such a way L ∩ V is cc-permutable in G and LV $\trianglelefteq$ G As L ≤ V this implies V ∩ LY = L(V ∩ Y) is cc-permutable in G. Then LY nearly c-permutable G and in light of (2) LY/L nearly c-permutable G/L.
Lemma 2.2.([12, Lemma 2.5]) Let q divides |G| in such a way (|G| , q2-1) = 1. Then G is q-nilpotent provided

G/X is q-nilpotent and q_3 does not divide |X|.
Lemma 2.3. ([3, Theorem 1.8.17]) Let $X \trianglelefteq G$ and X is solvable. Then Fit(X) is the direct product of some normal minimal abelian subgroups of G, provided $X \cap \Phi(G) = 1$.
Lemma 2.4. ([14, Lemma 2.8]) If N $syl_q \trianglelefteq G$ and L be a largest subgroup of G in such a way G = LN, Then:
(1) $(N \cap L) \trianglelefteq G$.
(2) If q > 2 then L has index q in G provided all smallest subgroups of N are normal in G.
Lemma 2.5. ([15, III, 3.5]) Assume that $X \le \Phi(G)$ with X, $Y \trianglelefteq G$. Implies Y is nilpotent provided $X \le Y$ and X/Y is nilpotent.
Lemma 2.6. ([14, p. 34]) Class of all p-closed groups is always a saturated formation provided p is a prime.
Lemma 2.7.([2]) Suppose that a saturated formation is denoted by F having all supersovable subgroups and $Y \trianglelefteq G$ such a way $G/Y \in F$. Then $G \in F$ provided Y is cyclic.
Lemma 2.8. ([18, Lemma 2.17])
(1) $Fit^*(X) \le Fit^*(G)$ provided $X \trianglelefteq G$.
(2) $Fit^*(G)/X \le Fit^*(G/X)$ provided $X \le Fit^*(G)$ and $X \trianglelefteq G$.
(3) $Fit(G) \le Fit^*(G) = Fit^*(Fit^*(G))$.
(4) $Fit(G) = Fit^*(G)$ provided $Fit^*(G)$ is solvable.
(5) $C_G(Fit^*(G)) \le Fit(G)$.
(6) If B(G) is the layer of G then $Fit^*(G) = B(G)Fit(G)$ and $B(G) \cap Fit(G) = Z(B(G))$([16, p. 128]).
Lemma 2.9.([4, Lemma 2.3(6)].) If $Q \trianglelefteq G$, implies $Fit^*(G/\Phi(Q)) = Fit^*(G)/\Phi(Q)$.
Proof: The proof of above Lemma is clearly a corollary of [16, X, (13.6)]
Lemma 2.11.([18, Lemma 2.20]) Let B be a q′- group of Aut(Q), where Q be a q-group of odd order. Then B is cyclic provided every subgroup of Q with prime order is B-invariant.
Lemma 2.12. ([19, Theorem A]) If X nearly c-permutable G and X is a q-group, then $O^q(G) \le N_G(X)$.

The proof of Theorem 1.3.
Proof. First we suppose our hypothesis is not true and assume that (G,X) be a counter example of smallest order. Let $Fit(X) = F_X$ and $Fit^*(X) = F^*_X$.
Now we suppose there exists smallest prime q dividing $|F_X|$ provided X is solvable. And if X is not solvable then q be the greatest prime divisor of $|F_X|$.
Suppose that Q $sylq \le F_X$ $Q_0 = \Omega_1(Q)$ and $C_G(Q_0) = C$. This is obvious that $C \trianglelefteq G$.
(1) $F^*_X = F_X$ X and $C_G(F_X) = C_G(F^*_X) \le F_X$.
Our theorem holds for (F^*_X, F^*_X) using Lemma 2.1, then F^*_X is supersolvable.
Hence by Lemma 2.8, $F_X = F^*_X$. This implies $G \in F$, provided $F_X = X$, a contradiction. So $F^*_X = F_X$ X. Also using Lemma 2.8,

$$C_G(F_X) = C_G(F^*_X) \le F_X.$$

(2) If Y *a* G and F_X is contained in Y then Y is supersolvable.
Here

$$F^*(Y) \le F^*_X = F_X \le Y$$

using Lemma 2.8, and it follows $F^*_X = F^*(Y)$. Thus by our choice of G, Y is supersolvable.
(3) X is supersoluble provided X = G. It implies directly by (2).
(4) Let X is soluble and $F_X(X/Q) = U/Q$ and P $sylp \le U$ provided p divides the order of U/Q. So p ƒ = q and either $p < q$ and $C_Q(Q) = 1$ or P F_X.
As U/Q is nilpotent and PQ/Q $sylp \le U/Q$, so PQ/Q is characteristic in U/Q. This implies $PQ \trianglelefteq X$. Hence p = q and PQ is supersoluble. Let p > q. Then PQ contained normal subgroup P and so $P \le F_X = F_X(X)$. Now assume q > p and $|F_X|$ has smallest prime divisor q, implies F_X is p′-subgroup. Let V $sylr \le F_X$ where $q \ne r$. Then p = r and so $[V,P] \le Q$. We get $y \in C_X(Q)$ for some $y \in P$. Since U/Q is nilpotent, so by [13, 5, Theorem 3.6],

$$[V, (y)] = [V, (y), (y)] = 1$$

implies $y \in C_G(F_X)$. As $C_X(F_X) \le F_X$ by (1) Hence $C_P(Q) = 1$.

(5) $p > 2$.

Let X is soluble and q = 2. Then (4) implies $F_X/Q = F(X/Q)$.
Also by Lemma 2.10 and (1),

$$F^*(X/Q) = F(X/Q) = F^*_X/Q.$$

So our hypothesis satisfied for (G/Q, X/Q) by Lemma 2.1. As

$$G/X \subset (G/Q)(X/Q) \in \mathbf{F}$$

Subsequently $G/Q \in \mathbf{F}$ and G also belongs to **F**. Which is a contradiction. Hence X is non solvable. This shows that p is the greatest prime divisor of order of F_X. With the help of (1), $F_X = F^*_X$ is a p-group where p = 2. Let $P \le X$ and $|P| = p$, where p 2 and let $Y = F_X P$. As Y is supersoluble implies $P \trianglelefteq Y$. So $P \le C_X(F_X)$, a contradiction to (1). Hence p > 2.
(6) There will be a smallest subgroup of Q which is not nearly c-permutable in G.
Here we may consider, every smallest subgroup of Q nearly c-permutable G. Let X is solvable and U/Q = F(X/Q) and P $sylp \le U$ and p divides the order of U/Q. So using (4) either $C_P(Q) = 1$ or $P \le F_X$. If $C_P(Q) = 1$ then P is cylic, by Lemma 2.11 and (5). Therefore our hypothesis is satisfied for G/Q by Lemma 2.1, it follows $G/Q \in F$ and hence $G \in F$. Which is a contradiction, so X is not solvable. Also by (3) X = G. Now we will prove that every smallest subgroup M of Q is normal in G. So first we show $O^q(G) = G$.
Let $O^q(G) = G$. Then in view of Lemma 2.8,

$$F^*(O^q(G)) \leq F^*_X$$

Thus

$$F^*(O^q(G)) = F^*_X \cap O^q(G) = F_X \cap O^q(G).$$

Then using Lemma 2.1 and (5) hypothesis is satisfied for $(O^q(G),O^q(G))$. Thus $O^q(G)$ is supersolvable, implies G is soluble, it follows that X is also solvable, which is a contradiction. So we can say $G = O^q(G)$ and using Lemma 2.12, $G = O^q(G) \leq N_G(M)$. Which shows that $M \trianglelefteq G$ and $Q_0 \leq Z(F_X)$. Since G/C is supersoluble by Lemma 2.1, so

$$(G/Q_0)/(C/Q_0) \text{ c } G/C \in \mathbf{F}.$$

Obviously in the light of Lemma 2.8,

$$F^*_X = F_X \leq F^*(C) \text{ and } F^*(C) \leq F^*_X$$

Therefore $F^*(C) = F^*_{x}$. As $Q_0 \leq Z(C)$,
it follows by Lemma 2.10

$$F^*(C/Q_0) = F^*_x/Q_0 = F_X/Q_0$$

So in view of Lemma 2.1, 2.3 and (5) hypothesis is satisfied for G/Q_0 implies

$G/Q_0 \in F$. But $Q_0 \leq Z^u$ follows by Lemma 2.7 $G \in \mathbf{F}$, a contradiction. Hence (6) holds (7) Q is not cyclic.

Using (6) we can say directly that Q is not cyclic.

(8) K is q-nilpotent provided K < G.

Since $|X_q| > q_2$ by lemma 2.2. Take K < G. As

$$K/(K \cap X) \cong KX/X \leq G/X$$

implies $K/(K \cap X)$ is q-nilpotent. We see by lemma 2.2, K is q-nilpotent provided $|K \cap X|q \leq q_2$. And using Lemma 2.1, every subgroup of $K \cap X$ of order q_2 is nearly c-permutable in K provided $|K \cap X|q > q_2$. Thus by our choice of G, K is q-nilpotent.

(9) G has the preceedings characteris tics:

(i) $QP \trianglelefteq G$, where $Q = G^{\mathfrak{R}}$ syl$q \trianglelefteq G$ and P be a non-normal cyclic Syl_p subgroup of G.

(ii) $\Phi(Q) \leq Z(Q)$.

(iii) If p = 2, then power of Q is 2 or 4, and if q > 2, then power of Q is p.

(iv) $Q/\Phi(Q)$ is smallest $\trianglelefteq G/\Phi(Q)$.

(v) |Q| is divisible by q_3.

(vi) $Q \leq X$.

Since G is minimal non-nilpotent by using (8) and [8, Theorem IV.5.4]. Hence in the light of [3, Theorem 3.4.2], (i)-(iv) are satisfied. And (6) is true by Lemma 2.2.Since G/X is q-nilpotent and $Q = G^{\mathfrak{R}}$ is the q-nilpotent residual of G.

(10) X is soluble.

Let X conatins a proper subgroup Y. Then Y/Y is supersoluble and |Y| < |G|. Let (v) be prime order or order 4 cyclic subgroup of any non-cyclic Syl_p Y.

So obviously v is also prime order or order 4 cyclic subgroup of a non-cyclic $Syl_p \leq X$. Thus (v) nearly c-permutable G by hypothesis. So by Lemma 2.1 (v) is nearly c-permutable in Y. Thus our hy- pothesis holds for (Y,Y). It follows Y is supersolvable by our selection of (G,X). Hence by [3, Theorem 3.11.8]) X is solvable.

(11) If $X \leq Q$ of order q_2, then X is cc-permutable.

Since X nearly c-permutable G. Then there will be $R \leq G$ in such a way $X \cap R$ is cc-permutable in G and $XR \trianglelefteq G$. If R < G, then R is nilpotent by (9). Again in view of (9) q_3 |Q| , q |R|. Suppose that R_q be a $Syl_q \leq R$. Implies $R_q \trianglelefteq R$ and $R \leq N_G(R_q)$. *As* $|X| = q_2$, $N_G(R_q) \trianglelefteq G$ *or* $|G:N_G(R_q)| = q$ or $|G:N_G(R_q)| = q_2$. Let $N_G(R_q) \trianglelefteq G$, then $R_q \trianglelefteq G$. Clearly, $q_3 | f | G/R_q|$ and $(G/R_q)/(G/R_q) = 1$ is q-nilpotent.

With the help of Lemma 2.2, G/R_q is q-nilpotent. Then $Q \leq R_q$ implies $Q = R_q$. This implies G = R, a contradiction. Now we suppose $|G:N_G(R_q)| = q$ and $Q_1 = Q \cap N_G(R_q)$. Since $|Q:Q_1| = |Q:(Q \cap N_G(R_q))| = |QN_G(R_q):N_G(R_q)| = |QR:N_G(R_q)| = |G:N_G(R_q)| = q$, Q_1 a largest subgroup of Q, so $Q_1 \trianglelefteq Q$. This implies $Q_1 \trianglelefteq G = Q\, N_G(R_q)$. If $Q_1 \subseteq \Phi(Q)$, then $Q = Q \cap X\, N_G = (R_q) = X(Q \cap N_G(R_q)) = X\, Q_1 = X$, a contradiction. Thus $Q_1\, f\!\subseteq \Phi(Q)$. As $Q/\Phi(Q)$ is the smallest normal subgroup of $G/\Phi(Q)$ implies $Q_1\Phi(Q)/\Phi(Q) = Q/\Phi(Q)$. It follows $Q = Q_1$, again contradiction. So we suppose $|G:N_G(R_q)| = q_2$. As $q_2 = |G:N_G(R_q)| \leq |G:R| = |XR:R| = |X:(X \cap R)| \leq q_2$, implies $X \cap R = 1$.

Thus, $|Q:R_q| = |XR_q:R_q| = |X:(XR_q)| = q_2$,
it follows that R_q is a 2- maximal subgroup of Q. So there will be a largest subgroup Q2 of Q in such a way R_q is the largest sungroup of Q2. Hence $R_q \trianglelefteq Q_2$ and $Q_2 \leq N_G(R_q)$. Then $|G:N_G(R_q)| = |XR:N_G(R_q)| = |QN_G(R_q):N_G(R_q)| = |Q:(Q \cap N_G(R_q))| \leq |Q:Q2| = q$, a contradiction. These contradictions shows R = G. Hence $X = X \cap R$ is cc-permutable in G.

(12) There exist $Y \leq Q$ in such a way $|Y| = q_2$ and $\Phi(Q)$ does not contains Y.

If $\Phi(Q) = 1$, then it is obvious. So Take $\Phi(Q) f = 1$.

If $|Q| = q_3$, then Q has a largest subgroup of order q_2. In the light of (7), Q is not cyclic. So Q must have atleast two largest subgroups Q_1 and Q_2. If $\Phi(Q)$ does contains Q_1 and Q_2, then $Q = Q_1 Q_2 \subseteq \Phi(Q)$, which is false. So we may suppose that $|Q| > q_3$. Assume that $y \in Q/\Phi(Q)$ and $b \in \Phi(Q)$ where |b| = q. As $\Phi(Q) \leq Z(Q)$, implies(y) (b) $\leq$ G. So |y| = q or 4 by (9). Now If |y| = 4, then we may select Y = (y). And if |y| = q, then | (y) (b) | $\leq q_2$. If | (y) (b) | = q, then (y) = (b), a contradiction. Hence | (y) (b) | = q_2.

(13) Our hypothesis holds for (A,A) and for (G/A,X/A) provided $A \trianglelefteq_{Hall} X$.

Let A contains a non-cyclic Syl_q sub- group Q. Then every cyclic subgroup M of Q nearly c-permutable G, where |M| is prime or 4. So M is nearly c- permutable in A. Thus hypothesis holds for (A,A).

Clearly $(G/A)/(X/A) \in \mathbf{F}$. Now if s divides |X/Q| and S

$syl_s \leq S^*$, we can say S^*/A be a non-cyclic $Syl_s \leq X/A$ in such a way $S^* = SA$. Then S is a non-cyclic $Syl_s \leq X$. Suppose that S^*/A contains a cyclic subgroup V/A of order 4 or prime order. As $A \trianglelefteq_{Hall} X$ so V/A = (y) A/A, where (y) is a subgroup of S of order 4 or prime order. By our supposition, y is nearly c-permutable in G. In the light of Lemma 2.1 V/A is also nearly c-permutable in G/A. So our hypothesis is satisfied for (G/A,X/A).

(14) X = S provided $S \trianglelefteq$ Halls X.

As S char X implies S is normal in G. So in the light of (13) our hypothesis holds for (G/S , X/S). Thus $G/S \in F$. It follows hypothesis holds for (G,S). Since G is of minimal order so X = S.

(15) Final Contradiction

By (9) $[Q]P \trianglelefteq G$. In view of (12), Q contains a subgroup X of order q^2, in such a way $Xf \leq \Phi(Q)$. This implies X is completely c-permutable in G by (11).

Then $XP^g = P^gX$, for some $g \in (X,P)$.

So

$X = X(P^g \cap Q) = XP^g \cap Q \trianglelefteq Xp^g$

This implies $P^g N_G(X)$. Also as $Q/\Phi(Q)$ is abelian, it follows

$X\Phi(Q)/\Phi(Q) \trianglelefteq Q/\Phi(Q)$.

Consequently

$X\Phi(Q)/\Phi(Q) \trianglelefteq G/\Phi(Q)$.

But $Q/\Phi(Q)$ is chief factor of G, so $X\Phi(Q)$ = Q implies X = Q. Which is not true.

Hence proved.

V. Some Applications

Corollary 5.1. ([11, Theorem 3]) Suppose that a saturated formation is denoted by F, having all the supersolvable groups and a solvable $H \trianglelefteq G$ in such a way G / H belongs to F. Then $G \in F$ provided all prime order sybgroups with order four of Fit(H) c-norm G.

Corollary 5.2. ([14, Theorem 4.1]) Suppose that a saturated formation is denoted by F, having all the supersolvable groups and a solvable $H \trianglelefteq G$ in such a way G / H belongs to F. Then $G \in F$ provided all cyclic subgroups and minimal subgroup of order four of Fit(H) c-supp G.

Corollary 5.3. ([14, Theorem 4.5]) Suppose that a saturated formation is denoted by F, having all the supersolvable groups and a solvable $H \trianglelefteq G$ in such a way $G / H \in F$. Subsequently $G \in F$ provided largest Syl_p subgroup of Fit(H) c-supp G.

Corollary 5.4. ([7, Theorem 1.6]) Suppose that a saturated formation is denoted by F, having all the supersolv- able groups and a solvable $H \trianglelefteq G$ in such a way $G / H \in F$. Then $G \in F$ provided largest Syl_p subgroup of Fit(H) comp G.

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