Multi-Attribute Decision Making Based on Averaging Aggregation Operators for Picture Hesitant Fuzzy Sets

K. Ullah[1], Z. Ali[2], N. Jan[3], T. Mahmood[4], S. Maqsood[5]

[1]International Islamic University Islamabad
[2,3,4]Mathematics & Statistics Department, International Islamic University, Islamabad, Pakistan
[5]Industrial Engineering Department, University of Engineering and Technology Peshawar, Pakistan
[1]kifayat.phdma72@iiu.edu.pk

Abstract-In this work, we proposed the theory of picture hesitant fuzzy set (PHFS) as a generalization of picture fuzzy set (PFS) and hesitant fuzzy set (HFS). Some basic operations on PHFSs have been developed and their results are studied. These aggregation operators included picture hesitant fuzzy weighted averaging (PHFWA) operator, picture hesitant fuzzy ordered weighted averaging (PHFOWA) operator and picture hesitant fuzzy hybrid averaging (PHFHA) operator. The basic properties of defined aggregation operators have been examined and some important results are discussed. Multi-Attribute decision making (MADM) process is discussed in the environment of PHFSs and numerical examples are provided for demonstration of proposed approach. The advantages of working in the environment of PHFSs are also discussed.

Keywords-Picture Fuzzy Set. Hesitant Fuzzy Set. Picture Hesitant Fuzzy Set. Aggregation Operators. Decision Making.

I. INTRODUCTION

Decision making (DM) is one of the most applied phenomena of fuzzy set (FS) theory which is applied to many real-life problems like in engineering, economics and management sciences etc. In recent decades, DM has been used in problems having impreciseness and uncertainties in the data on a large scale. In order to get best alternative among some set of alternatives or to get a suitable strategy from a list of strategies or any other circumstances where one needed to a take a decision, we need some mathematical tools and computing algorithms tools. Normally in our real life, in any problem of DM we face some sort of uncertainty. Such type of situations can be handled using the tools fuzzy sets (FSs) [i], intuitionistic fuzzy sets (IFSs) [ii], PFSs [iii] and HFSs [iv, v].

In 1965, the theory of FS [i] is iniated to deal with imprecise events. A FS is based on a membership function which assigned a membership grade to an element of a set on a scale of 0 to 1. A membership grade of 0 mean that element does not belong to that set while a membership grade of 1 means that element is fully belong to that set. Similarly, a membership grade between 0 to 1 represent the degree to which the element satisfies the property of certain set.

In the 1986, an enhanced extension of FS is proposed, known as IFS [ii]. A non-membership function is included in FS which generalizes it and the new structure is named as IFS. The membership and non-membership functions provide the degree of affiliation and disaffiliation of an element. In 2013, a new concept of PFS [iii] is proposed which is based on four membership grades. A PFS is a generalization of FS and IFS which not only described of membership and non-membership but also the abstinence as well as refusal degree of an element. PFS is a strong concept compared to FS and IFS and can be applied in many problems.

While modeling uncertain situations, V. Torra and Y. Narukawa presented the concept of HFS. In HFS the membership of an element is described in term of few elements in [iv]. A HFS is a direct generalization of FS and have been applied to real life problems extensively. Based on several different situations, some new extensions of FS, IFS and HFS have been proposed so for including IHFS [vi], hesitant bi-polar fuzzy aggregation operator in DM [vii], interval valued HFS [viii], interval valued IHFS [ix], bi-polar valued HFS [x] etc. These advanced structures have been used extensively so far in real-life problems.

Recently, the potential work on PFSs and its advanced forms such as Spherical fuzzy set and T-spherical fuzzy sets have been done. For example, the idea of T-spherical fuzzy sets and its applications in MADM are discussed in [xi]. Further, some similarity measures for T-Spherical fuzzy sets are developed in [xii]. Some novel operations on relations of PFSs are developed in [xiii]. The graphs of PFSs are proposed in [xiv] which were further applied to practical situations.

FS is applied in MADM problem in [xv, xvi] while the notion of IFSs are utilized in decision making problems by [xvii]. MADM based on averaging and geometric aggregation operators of IFSs are studied by [xviii, xix]. The Einstein intuitionistic fuzzy aggregation operators are and their applications in

MADM are studied in [xx]. Some weighted averaging operators are developed by [xxi] while weighted geometric aggregation operators are developed in [xxii] for PFSs which were further utilized in MADM. The concept of cubic linguistic fuzzy sets has been utilized in MADM by [xxiii]. Some aggregations tools for bipolar valued HFSs are developed in [xxiv, xxv] and their viability is studied.The idea of cross entropy for PFSs is utilized in MADM by [xxvi]. An algorithm for MADM based on interaction aggregation operators of T-spherical fuzzy sets are developed in [xxvii]. In [xxviii], the concept of linguistic Pythagorean FS is applied to MADM problems while in [xxix] the decision-making problems are solved using hesitant Pythagorean FSs. Some generalized interactive aggregation operators of Pythagorean FSs are proposed in [xxx] and applied in MADM problems. In [xxxi], TOPSIS approach is followed to solve decision-making problems. Some other quality work in this regard can be found in [xxxii-xxxv].

In this article, we proposed a generalization of PFS by combining PFS and HFS known as PHFS. This type of structures can be useful when the values of membership, neutral and non-membership are in the form of HFSs. This type of new structure could be adequate to model real life challenges. This article starts with introduction as its first section. In section two, some basic ideas of IFS, PFS and HFS are described. Section three is based on PHFS and its basic operations. In section four, some aggregation operators are developed for PHFSs and their basic properties are investigated. In section five DM process is described in the environment of PHFS and illustrated with an example. The article ends with some advantages of proposed work and concluding remarks.

II. PRELIMINARIES

This section includes some very basic definitions related to IFSs, HFSs and PFSs. The set theoretic operations of these structures are also included in this section. The work done discussed in this section provides a foundation for the proposed work.

Definition 1: [ii] On a universe of discourse *X*, an IFS is of the shape

$$A = \{< x, \mu_A(x), \eta_A(x) >:\ x \in X\}$$

Here $\mu_A, \eta_A : X \to [0,1]$ denotes degree of satisfaction and dissatisfaction provided that $0 \leq \mu_A(x) + \eta_A(x) \leq 1$. Further $\pi_A(x) = 1 - \left(\mu_A(x) + \eta_A(x)\right)$ is called the degree of hesitancy and (μ_A, η_A) is called intuitionistic fuzzy number (IFN).

Definition 2: [ii] For two IFNs $A = (\mu_A, \eta_A)$ and $B = (\mu_B, \eta_B)$ their basic operations are defined as:

(1) $\mu_A \leq \mu_B$ and $\eta_A \geq \eta_B$ implies $A \sqsubset B$.

(2) $A \sqsubseteq B$ and $A \sqsupseteq B$ implies $A = B$.

(3) $A^c = (\eta_A, \mu_A)$

(4) $A \cap B = \left(min(\mu_A, \mu_B), max(\eta_A, \eta_B)\right)$

(5) $A \cup B = \left(max(\mu_A, \mu_B), min(\eta_A, \eta_B)\right)$

Definition 3: [iii] On a universe of discourse *X*, a PFS is of the shape

$$A = \{< x, \mu_A(x), \eta_A(x), \xi_A(x) >:\ x \in X\}$$

Here $\mu_A, \eta_A, \xi_A : X \to [0,1]$ denotes degree of satisfaction, abstinence and dissatisfaction provided that $0 \leq \mu_A(x) + \eta_A(x) + \xi_A(x) \leq 1$. Further $\pi_A(x) = 1 - \left(\mu_A(x) + \eta_A(x) + \xi_A(x)\right)$ is called the degree of hesitancy and (μ_A, η_A, ξ_A) is called picture fuzzy number (PFN).

Definition 4: [iii] For two PFNs $A = (\mu_A, \xi_A, \eta_A)$ and $B = (\mu_B, \xi_B, \eta_B)$ their basic operations are defined as:

(1) $\mu_A \leq \mu_B$, $\xi_A \geq \xi_B$ and $\eta_A \geq \eta_B$ implies $A \sqsubset B$.

(2) $A \sqsubseteq B$ and $A \sqsupseteq B$ implies $A = B$.

(3) $A \cap B = \left(min(\mu_A, \mu_B), min(\xi_A, \xi_B), max(\eta_A, \eta_B)\right)$

(4) $A \cup B = \left(max(\mu_A, \mu_B), min(\xi_A, \xi_B), min(\eta_A, \eta_B)\right)$

(5) $A^C = (\eta_A, \xi_A, \mu_A)$.

Definition 5: [v] A HFS in terms of a membership function gives us few elements from [0,1] against each $x \in X$. i.e. $H = \{< x, \mu(x) > : \mu(x)$ is a finite subset of $[0,1]\ \forall\ x \in X\}$.

Moreover $\mu(x)$ is called hesitant fuzzy number (HFN).

Definition 6: [xxi] A mapping T: [0,1] x [0,1] $\to$ [0,1] is denoted t-norm if T is satisfy the following conditions:

(1) Boundary condition.
(2) Monotonicity
(3) Commutative
(4) Associativity

On the other hand, a mapping K is denoted t-conorm if $K(a,b) = 1 - T(1-a, 1-b)$ for all $a, b \in$ [0,1].

Definition 7: [xxi] An Archimedean t-norm (t-conorm) mapping T (or K) is a continuous t-norm (t-conorm) satisfying the conditions $T(a,a) < a$ (or $K(a,a) > a$) for all $a \in [0,1]$.

Strict Archimedean t-norm T and t-conorm K can be expressed using continuous mapping h, g: [0,1] $\to$ [0, ∞), then $T(a,b) = g^{-1}\left(g(a) + g(b)\right)$ and $K(a,b) = h^{-1}\left(h(a) + h(b)\right)$ where h, g is decreasing and increasing function with $g(1) = 0$ and $h(0) = 0$, where $g(a) = h(1-a)$.

III. Picture Hesitant Fuzzy Set

This section is based on the concepts of PHFSs and their basic operations. Few examples are provided in support of our work.
First, the definition of PHFS is proposed.

Definition 8: On a universe of discourse X, a PHFS is of the shape

$$A = \{< x, \mu_A(x), \eta_A(x), \xi_A(x) >:\ x \in X\}$$

where μ, ξ and η are HFNs denoting the satisfaction, abstinence and dissatisfaction degree provided that $0 \leq \sup(\mu_A(x)) + \sup(\xi_A(x)) + \sup(\eta_A(x)) \leq 1$ and $\delta_A(x) = 1 - (\sup(\mu_A(x)) + \sup(\xi_A(x)) + \sup(\eta_A(x)))$ is denoted the refusal degree. Moreover, (μ, ξ, η) is called PHFN.

Example 1: Let $X = \{1,2,3\}$. Then a PHFS on X is defined as:

P

$= (<1, (\{0.4,0.5\}, \{0.3,0.1\}, \{0.1,0.2\})>$
$<2, (\{0.3,0.3\}, \{0.5,0.4\}, \{0.2,0.0\})>,$
$<3, (\{0.2,0.0\}, \{0.2,0.4\}, \{0.3,0.4\})>)$

Definition 9: Let P_1 and P_2 be the two PHFNs. Then

(1) $P_1 \cup P_2 = (max(\mu_1, \mu_2), min(\xi_1, \xi_2), min(\eta_1, \eta_2),)$

(2) $P_1 \cap P_2 = (min(\mu_1, \mu_2), min(\xi_1, \xi_2), max(\eta_1, \eta_2),)$

(3) $P_1^c = (\eta_1, \xi_2, \mu_1)$

Example 2: Let
$P_1 = (\{0.1,0.4\}, \{0.3,0.2\}, \{0.2,0.1\})$ and
$P_2 = (\{0.3,0.1\}, \{0.3,0.4\}, \{0,0.1\})$ be two PHFNs. Then

(1) $P_1 \cup P_2 =$

$$\begin{pmatrix} \max(\{0.1,0.4\}, \{0.3,0.1\}), \\ \min(\{0.3,0.2\}, \{0.3,0.4\}), \\ \min(\{0.2,0.1\}, \{0,0.1\}), \end{pmatrix}$$

$= (\{0.3,0.4\}, \{0.3,0.2\}, \{0.2,0.1\})$

(2) $P_1 \cap P_2 =$

$$\begin{pmatrix} \min(\{0.1,0.4\}, \{0.3,0.1\}), \\ \min(\{0.3,0.2\}, \{0.3,0.4\}), \\ \max(\{0.2,0.1\}, \{0,0.1\}), \end{pmatrix}$$

$= (\{0.1,0.1\}, \{0.3,0.2\}, \{0,0.1\})$

(3) $P_1^c = (\{0.2,0.1\}, \{0.3,0.2\}, \{0.1,0.4\})$

Theorem 1: Let P_1, P_2 and P_3 be three PHFNs. Then

(1) $P_1 \cap P_2 = P_2 \cap P_1$

(2) $P_1 \cup P_2 = P_2 \cup P_1$

(3) $(P_1 \cap P_2) \cap P_3 = P_1 \cap (P_2 \cap P_3)$

(4) $(P_1 \cup P_2) \cup P_3 = P_1 \cup (P_2 \cup P_3)$

(5) $(P_1^c)^c = P_1$

(6) $(P_1 \cap P_2)^c = {P_1}^c \cap {P_2}^c$

(7) $(P_1 \cup P_2)^c = {P_1}^c \cup {P_2}^c$

Proof: Proof is straightforward.

To rank PHFNs (whenever necessary), we need to define score and accuracy function for PHFNs.

Definition 10: The score value of a PHFN $P = (\mu,\xi,\eta)$ is defined as:

$$S(P) = \frac{\hat{S}(\mu) - \hat{S}(\xi) - \hat{S}(\eta)}{3} \qquad (1)$$

Where

$$\hat{S}(\mu) = \frac{sum\ of\ all\ elements\ in\ \mu}{order\ of\ \mu}$$

$$\hat{S}(\xi) = \frac{sum\ of\ all\ elements\ in\ \xi}{order\ of\ \xi}$$

$$\hat{S}(\eta) = \frac{sum\ of\ all\ elements\ in\ \eta}{order\ of\ \eta}$$

Remark 1: For two PHFNs P_1 and P_2. If

(1) $S(P_1) > S(P_2)$. Then P_1 is superior than P_2

(2) $S(P_1) < S(P_2)$. Then P_1 is inferior than P_2

(3) $S(P_1) = S(P_2)$. Then P_1 and P_2 are similar

Example 3: Consider two PHFNs $P_1 = (\{0.3,0.2\}, \{0.1,0.3\}, \{0.2,0.1\})$ and $P_2 = (\{0.2,0.1\}, \{0.0,0.2\}, \{0.1,0.1\})$.
Then $S(P_1) = -0.033$ and $S(P_2) = -0.0167$. As $S(P_2) > S(P_1)$ so $P_2 > P_1$.

Sometime, we could not rank PHFNs based on score values. For instance, if
$P_1 = (\{0.7,0.1\}, \{0.1,0.2\}, \{0.1,0.1\})$ and
$P_2 = (\{0.5,0.1\}, \{0.1,0.1\}, \{0.1,0.0\})$ Then it is impossible to rank them
because $S(P_1) = S(P_2) = 0.05$. To differentiate between them, we defined accuracy function.

Definition 11: The accuracy value of a PHFN $P = (\mu,\xi,\eta)$ is defined as:

$$H(P) = \frac{\hat{S}(\mu) + \hat{S}(\xi) + \hat{S}(\eta)}{3} \qquad (2)$$

Where $\hat{S}(\mu)$, $\hat{S}(\xi)$ and $\hat{S}(\eta)$ can be calculated as in definition 8.

Example 4: Let us consider the above case where $P_1 = (\{0.7,0.1\}, \{0.1,0.2\}, \{0.1,0.1\})$ and $P_2 = (\{0.5,0.1\}, \{0.1,0.1\}, \{0.1,0.0\})$. Then $S(P_1) = S(P_2)$. By calculating the accuracy value, we get $H(P_1) = 0.216 > H(P_2) = 0.15$. Hence $P_1 > P_2$.

IV. AGGREGATION OPERATORS

The aim of this section is to introduce some novel aggregation operators for PHFSs such as PHFWA, PHFOWA and PHFHA operators. The mentioned operators are not only developed in this section but also their characteristics have been studied and their fitness is established using induction phenomenon. Later, in next section these proposed operators are applied in MADM problem.

Definition 12: Let $P = (\mu,\xi,\eta)$, $P_1 = (\mu_1,\xi_1,\eta_1)$ and $P_2 = (\mu_2,\xi_2,\eta_2)$ be three PHFNs. Then

$$P_1 \oplus P_2 = \left(h^{-1}\left(h(\mu_1) + h(\mu_2)\right), g^{-1}\left(g(\xi_1) + g(\xi_2)\right), g^{-1}\left(g(\eta_1) + g(\eta_2)\right)\right)$$

(1) $$P_1 \otimes P_2 = \left(g^{-1}\left(g(\mu_1) + g(\mu_2)\right), h^{-1}\left(h(\xi_1) + h(\xi_2)\right), h^{-1}\left(h(\eta_1) + h(\eta_2)\right)\right)$$

(2) $$\lambda P = \begin{pmatrix} h^{-1}\left(\lambda h(\mu)\right), g^{-1}\left(\lambda g(\xi)\right), \\ g^{-1}\left(\lambda g(\eta)\right) \end{pmatrix}$$

(3) $$P^{\lambda} = \begin{pmatrix} g^{-1}\left(\lambda g(\mu)\right), h^{-1}\left(\lambda h(\xi)\right), \\ h^{-1}\left(\lambda h(\eta)\right) \end{pmatrix}$$

Theorem 2: Let P_1, P_2 and P_3 be three PHFNs and λ, λ_1 and $\lambda_2 > 0$. Then

(a) $P_1 \oplus P_2 = P_2 \oplus P_1$

(b) $P_1 \otimes P_2 = P_2 \otimes P_1$

(c) $\lambda(P_1 \oplus P_2) = \lambda P_1 \oplus \lambda P_2$

(d) $(P_1 \otimes P_2)^{\lambda} = {P_1}^{\lambda} \otimes {P_2}^{\lambda}$

(e) $\lambda_1 P \oplus \lambda_2 P = (\lambda_1 + \lambda_2)P$

(f) $P^{\lambda_1} \otimes P^{\lambda_1} = P^{\lambda_1+\lambda_2}$

Proof: Proof is straightforward.

In our next study, P_i and Q_k will denote collection of PHFNs $w_i = (w_1, w_2, ...w_n)$ will denote the weight vector such that $w_i > 0$ and $\sum_{i=1}^{n} w_i = 1$ where $i,k,j = 1,2,3,...n$.

Definition 13: For a collection of PHFNs P_i, the PHFWA operator is defined as:

$$PHFWA(P_1, P_2, \dots, P_n) = w_1P_1 \oplus w_2P_2 \oplus \dots \dots . \oplus w_nP_n = \oplus_{i=1}^{n} w_iP_i$$

Theorem 3: For a collection of PHFNs P_i, their aggregated value is a PHFN and can be obtained as:

$$PHFWA(P_1, P_2, ...P_n) = \begin{pmatrix} h^{-1}\sum_{i=1}^{n} w_i h(\mu_i), g^{-1}\sum_{i=1}^{n} w_i g(\xi_i), \\ g^{-1}\sum_{i=1}^{n} w_i g(\eta_i) \end{pmatrix} \quad (3)$$

Proof: This result is proved by using principle of mathematical induction.

Case 1: For $n = 2$ we have $P_1 = (\mu_1,\xi_1,\eta_1)$, $P_2 = (\mu_2,\xi_2,\eta_2)$ and

$$w_1P_1 = \begin{pmatrix} h^{-1}\left(w_1h(\mu_1)\right), g^{-1}\left(w_1g(\xi_1)\right), \\ g^{-1}\left(w_1g(\eta_1)\right) \end{pmatrix}$$

$$w_2P_2 = \begin{pmatrix} h^{-1}\left(w_2h(\mu_2)\right), g^{-1}\left(w_2g(\xi_2)\right), \\ g^{-1}\left(w_2g(\eta_2)\right) \end{pmatrix}$$

Hence by additive law of PHFSs, we get

$$PHFWA(P_1, P_2) = w_1P_1 \oplus w_2P_2$$

$$= \begin{pmatrix} h^{-1}\left(w_1h(\mu_1)\right), \\ g^{-1}\left(w_1g(\xi_1)\right), \\ g^{-1}\left(w_1g(\eta_1)\right) \end{pmatrix} \oplus \begin{pmatrix} h^{-1}\left(w_2h(\mu_2)\right), \\ g^{-1}\left(w_2g(\xi_2)\right), \\ g^{-1}\left(w_2g(\eta_2)\right) \end{pmatrix}$$

$$= \begin{pmatrix} h^{-1}\begin{pmatrix} h\left(h^{-1}\left(w_1h(\mu_1)\right)\right) + \\ h\left(h^{-1}\left(w_2h(\mu_2)\right)\right) \end{pmatrix}, \\ g^{-1}\begin{pmatrix} g\left(g^{-1}\left(w_1g(\xi_1)\right)\right) + \\ g\left(g^{-1}\left(w_2g(\xi_2)\right)\right) \end{pmatrix}, \\ g^{-1}\begin{pmatrix} g\left(g^{-1}\left(w_1g(\eta_1)\right)\right) + \\ g\left(g^{-1}\left(w_2g(\eta_2)\right)\right) \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} h^{-1}\left(\left(w_1h(\mu_1)\right) + \left(w_2h(\mu_2)\right)\right), \\ g^{-1}\left(w_1g(\xi_1)\right) + \left(w_2g(\xi_2)\right), \\ g^{-1}\left(\left(w_1g(\eta_1)\right) + \left(w_2g(\eta_2)\right)\right) \end{pmatrix}$$

$$= \begin{pmatrix} h^{-1}\sum_{i=1}^{2}\left(w_i h(\mu_i)\right), \\ g^{-1}\sum_{i=1}^{2}\left(w_i g(\xi_i)\right), \\ g^{-1}\sum_{i=1}^{2}\left(w_i g(\eta_i)\right) \end{pmatrix}$$

The result holds for $n = 2$.

Case 2: Assuming the result holds good for $n = k$. Then for $n = k + 1$, we have

$$PHFWA(P_1, P_2, \ldots, P_{K+1})$$
$$= \oplus_{i=1}^{K} w_i P_i \oplus w_{K+1} P_{K+1}$$

$$= \begin{pmatrix} h^{-1}\sum_{i=1}^{K}\left(w_i h(\mu_i)\right), \\ g^{-1}\sum_{i=1}^{K}\left(w_i g(\xi_i)\right), \\ g^{-1}\sum_{i=1}^{K}\left(w_i g(\eta_i)\right) \end{pmatrix} \oplus \begin{pmatrix} h^{-1}\left(w_{K+1} h(\mu_{k+1})\right), \\ g^{-1}\left(w_{K+1} g(\xi_{k+1})\right), \\ g^{-1}\left(w_{K+1} g(\eta_{k+1})\right) \end{pmatrix}$$

$$= \begin{pmatrix} h^{-1}\left\{h\begin{pmatrix} h^{-1}\sum_{i=1}^{K}\left(w_i h(\mu_i)\right) + \\ h^{-1}\left(w_{K+1} h(\mu_{k+1})\right) \end{pmatrix}\right\}, \\ g^{-1}\left\{g\begin{pmatrix} g^{-1}\sum_{i=1}^{K}\left(w_i g(\xi_i)\right) + \\ g^{-1}\left(w_{K+1} g(\xi_{k+1})\right) \end{pmatrix}\right\}, \\ g^{-1}\left\{g\begin{pmatrix} g^{-1}\sum_{i=1}^{K}\left(w_i g(\eta_i)\right) + \\ g^{-1}\left(w_{K+1} g(\eta_{k+1})\right) \end{pmatrix}\right\} \end{pmatrix}$$

$$= \begin{pmatrix} h^{-1}\left\{\sum_{i=1}^{K}\left(w_i h(\mu_i)\right) + w_{K+1} h(\mu_{k+1})\right\}, \\ g^{-1}\left\{\sum_{i=1}^{K}\left(w_i g(\xi_i)\right) + w_{K+1} g(\xi_{k+1})\right\}, \\ g^{-1}\left\{\sum_{i=1}^{K}\left(w_i g(\eta_i)\right) + w_{K+1} g(\eta_{k+1})\right\} \end{pmatrix}$$

$$= \begin{pmatrix} h^{-1}\left\{\sum_{i=1}^{K+1}\left(w_i h(\mu_i)\right)\right\}, \\ g^{-1}\left\{\sum_{i=1}^{K+1}\left(w_i g(\xi_i)\right)\right\}, \\ g^{-1}\left\{\sum_{i=1}^{K+1}\left(w_i g(\eta_i)\right)\right\} \end{pmatrix}$$

So, by induction hypothesis the result is true $\forall\, n \in \mathbb{Z}$.

Theorem 4: PHFWA operator in PHFSs becomes IHFWA operator If $\xi_i = 0$ for every I.

Proof: Using $h(t) = g(1 - t)$ (3) becomes PHFWA (P_1, P_2, ...P_n)

$$= \begin{pmatrix} 1 - g^{-1}\sum_{i=1}^{n}\left(w_i h(\mu_i)\right), \\ g^{-1}\sum_{i=1}^{n}\left(w_i g(\eta_i)\right) \end{pmatrix} \quad (4)$$

$$= \begin{pmatrix} 1 - g^{-1}\sum_{i=1}^{n}\left(w_i h(1-\mu_i)\right), \\ g^{-1}\sum_{i=1}^{n}\left(w_i g(\eta_i)\right) \end{pmatrix}$$

$$= \mathrm{I}HFWA(P_1, P_2, \ldots, P_n)$$

The above IHFWA operator in IHFSs is the modified form of PHFWA operator.

Example 5: Let
$P_1 = (\{0.5,0.2\}, \{0.1,0.3\}, \{0.2,0.2\})$,
$P_2 = (\{0.1,0.2\}, \{0.2,0.3\}, \{0.2,0.4\})$ and
$P_3 = (\{0.1,0.1\}, \{0.2,0.2\}, \{0.3,0.4\})$ be three PHFNs whose weight vector is $w = (0.2,0.3,0.5)^T$. Considering $g(l,q) = lq$ and $h(l,q) = 1 - (1 - l)(1 - q)$ in [xxxvi]. Then PHFWA (P_1,P_2,P_3) = ({0.20,0.14}, {0.18,0.25}, {0.25,0.34}). Which is clearly a PHFNs.

The PHFWA operator stated in theorem 3 satisfies the properties of idempotency, monotonicity, boundedness, shift invariance and homogeneity which are elaborated as follows.

Theorem 5 (Idempotency): If $P_i = P = (\mu,\xi,\eta)$ for all i, then PHFWA (P_1,P_2,...,P_n) = P.

Proof: As $\forall\, i\, P_i = P$. So

$PHFWA\,(P_1,P_2,\ldots,P_n)$

$$= \begin{pmatrix} h^{-1}\sum_{i=1}^{n}\left(w_i h(\mu_i)\right), g^{-1}\sum_{i=1}^{n}\left(w_i g(\xi_i)\right), \\ g^{-1}\sum_{i=1}^{n}\left(w_i g(\eta_i)\right) \end{pmatrix}$$

$$= \begin{pmatrix} h^{-1}\sum_{i=1}^{K}\left(w_i h(\mu)\right), g^{-1}\sum_{i=1}^{K}\left(w_i g(\xi)\right), \\ g^{-1}\sum_{i=1}^{K}\left(w_i g(\eta)\right) \end{pmatrix}$$

$$= \left(h^{-1}h(\mu), g^{-1}g(\xi), g^{-1}g(\eta)\right)$$
$$= (\mu, \xi, \eta) = P.$$

Theorem 6 (Monotonicity): For two PHFNs P_i and Q_k such that $P_i \leq Q_k$ for all j that is $\mu_{ij} \leq \mu_{kj}$, $\xi_{ij} \geq \xi_{kj}$ and $\eta_{ij} \geq \eta_{kj}$. Then

$$PHFWA(P_1, P_2, \ldots, P_n) \leq PHFWA(Q_1, Q_2, \ldots, Q_n)$$

Proof: Since $\mu_{ij} \leq \mu_{kj}$, $\xi_{ij} \geq \xi_{kj}$ and $\eta_{ij} \geq \eta_{kj}$ and g and h are monotonic functions (h is decreasing while g is increasing).

$$h^{-1}\sum_{j=1}^{n}\left(w_j h(\mu_{ij})\right) \leq h^{-1}\sum_{j=1}^{n}\left(w_j h(\mu_{kj})\right),$$

$$g^{-1}\sum_{j=1}^{n}\left(w_j g(\xi_{ij})\right) \geq g^{-1}\sum_{j=1}^{n}\left(w_j g(\xi_{kj})\right),$$

$$g^{-1}\sum_{j=1}^{n}\left(w_j g(\eta_{ij})\right) \geq g^{-1}\sum_{j=1}^{n}\left(w_j g(\eta_{kj})\right).$$

Therefore, by score function of the PHFNs, we get

$$S\left(PHFWA(P_1, P_2, \ldots, P_n)\right)$$

$$= \left(h^{-1}\sum_{j=1}^{n}\left(w_j h(\mu_{ij})\right) - g^{-1}\sum_{j=1}^{n}\left(w_j g(\xi_{ij})\right) - g^{-1}\sum_{j=1}^{n}\left(w_j g(\eta_{ij})\right) \right)$$

$$\leq \left(h^{-1}\sum_{j=1}^{n}\left(w_j h(\mu_{kj})\right) - g^{-1}\sum_{j=1}^{n}\left(w_j g(\xi_{kj})\right) - g^{-1}\sum_{j=1}^{n}\left(w_j g(\eta_{kj})\right) \right)$$

$$\leq S\left(PHFWA(Q_1, Q_2, \ldots, Q_n)\right)$$

Hence

$$S\left(PHFWA(P_1, P_2, \ldots, P_n)\right) \leq S\left(PHFWA(Q_1, Q_2, \ldots, Q_n)\right).$$

Theorem 7 (Boundedness): If $P^- = \left(\min(\mu_j), (\min(\xi_j), (\max(\eta_j)\right)$, and $P^+ = \left(\max(\mu_j), (\min(\xi_j), (\max(\eta_j)\right)$. Then

$$P^- \leq \left(PHFWA(P_1, P_2, \ldots, P_n)\right) \leq P^+$$

Proof: Since for all i, we have $min(\mu_j) \leq (\mu_j) \leq max(\mu_j)$, $max(\xi_j) \leq (\xi_j) \leq \min(\xi_j)$ and, $max(\eta_j) \leq (\eta_j) \leq min(\eta_j)$ and the generators g and h are monotonic functions (h is decreasing while g is increasing).

$$h^{-1}\sum_{j=1}^{n}\left(w_j h\ min(\mu_j)\right)$$

$$\leq h^{-1}\sum_{j=1}^{n}\left(w_j h(\mu_j)\right)$$

$$\leq h^{-1}\sum_{j=1}^{n}\left(w_j h(max(\mu_j)\right)$$

$$g^{-1}\sum_{j=1}^{n}\left(w_j g\ min(\xi_j)\right)$$

$$\leq g^{-1}\sum_{j=1}^{n}\left(w_j g(\xi_j)\right)$$

$$\leq g^{-1}\sum_{j=1}^{n}\left(w_j g\ max(\xi_j)\right)$$

$$g^{-1}\sum_{j=1}^{n}\left(w_j g\, min(\eta_j)\right)$$

$$\leq g^{-1}\sum_{j=1}^{n}\left(w_j g(\eta_j)\right)$$

$$\leq g^{-1}\sum_{j=1}^{n}\left(w_j g\, max(\eta_j)\right)$$

$$min(\mu_j) \leq h^{-1}\sum_{j=1}^{n}\left(w_j h(\mu_j)\right) \leq max(\mu_j),$$

$$min(\xi_j) \leq g^{-1}\sum_{j=1}^{n}\left(w_j g(\xi_j)\right) \leq max(\xi_j)$$

$$min(\eta_j) \leq g^{-1}\sum_{j=1}^{n}\left(w_j g(\eta_j)\right) \leq max(\eta_j)$$

Now, according to the note, we have

$$\left(max(\mu_j), max(\xi_j), max(\eta_j)\right)$$

$$\leq \left(h^{-1}\sum_{j=1}^{n}\left(w_j h(\mu_j)\right), g^{-1}\sum_{j=1}^{n}\left(w_j g(\xi_j)\right), g^{-1}\sum_{j=1}^{n}\left(w_j g(\eta_j)\right) \right)$$

$$\leq \left(min(\mu_j), min(\xi_j), min(\eta_j)\right)$$

Therefore

$$P^- \leq \begin{pmatrix} h^{-1}\sum_{j=1}^{n}\left(w_j h(\mu_j)\right), g^{-1}\sum_{j=1}^{n}\left(w_j g(\xi_j)\right), \\ g^{-1}\sum_{j=1}^{n}\left(w_j g(\eta_j)\right) \end{pmatrix} \leq P^+$$

i.e. $P^- \leq \left(PHFWA(P_1, P_2, \dots, P_n)\right) \leq P^+$

This completes the proof.

Theorem 8 (Shift Invariance): For some PHFNs P_i. We have

$PHFWA(P_1 \oplus P_1, P_2 \oplus P_1, \dots, P_n \oplus P_1)$

$= PHFWA(P_1, P_2, \dots, P_n) \oplus P_1$

Proof: As $P_i = (\mu_i, \xi_i, \eta_i)$ and $P_1 = (\mu_1, \xi_1, \eta_1)$ are PHFNs. Then $\forall_i$, we have

$P_i \oplus P_1 =$

$(h^{-1}\{h(\mu_i) + h(\mu_1)\}, g^{-1}\{g(\xi_i) + g(\xi_1)\}, g^{-1}\{g(\eta_i) + g(\eta_1)\})$

Therefore

$PHFWA(P_1 \oplus P_1, P_2 \oplus P_1, \dots, P_n \oplus P_1)$

$$= \begin{pmatrix} h^{-1}\sum_{i=1}^{n}\left(w_i h\{h^{-1}\{h(\mu_i) + h(\mu_1)\}\}\right), \\ g^{-1}\sum_{i=1}^{n}\left(w_i g\{g^{-1}\{g(\xi_i) + g(\xi_1)\}\}\right), \\ g^{-1}\sum_{i=1}^{n}\left(w_i g\{, g^{-1}\{g(\eta_i) + g(\eta_1)\}\}\right) \end{pmatrix}$$

$$= \begin{pmatrix} h^{-1}\sum_{i=1}^{n} w_i \{h(\mu_i) + h(\mu_1)\}, \\ g^{-1}\sum_{i=1}^{n} w_i \{g(\xi_i) + g(\xi_1)\}, \\ g^{-1}\sum_{i=1}^{n} w_i \{g(\eta_i) + g(\eta_1)\} \end{pmatrix}$$

$$= \begin{pmatrix} h^{-1}\left\{\sum_{i=1}^{n} w_i \left(h(\mu_i)\right) + h(\mu_1)\right\}, \\ g^{-1}\left\{\sum_{i=1}^{n} w_i \left(g(\xi_i)\right) + g(\xi_1)\right\}, \\ g^{-1}\left\{\sum_{i=1}^{n} w_i \left(g(\eta_i)\right) + g(\eta_1)\right\} \end{pmatrix}$$

$$= \begin{pmatrix} h^{-1}\left\{h\left\{h^{-1}\left\{\sum_{i=1}^{n} w_i \left(h(\mu_i) + h(\mu_1)\right)\right\}\right\}\right\}, \\ g^{-1}\left\{g\left\{g^{-1}\left\{\sum_{i=1}^{n} w_i \left(g(\xi_i) + g(\xi_1)\right)\right\}\right\}\right\}, \\ g^{-1}\left\{g\left\{g^{-1}\left\{\sum_{i=1}^{n} w_i \left(g(\eta_i) + g(\eta_1)\right)\right\}\right\}\right\} \end{pmatrix}$$

$$= \begin{pmatrix} h^{-1}\sum_{i=1}^{n} w_i \left(h(\mu_i)\right), \\ g^{-1}\sum_{i=1}^{n} w_i \left(g(\xi_i)\right), \\ g^{-1}\sum_{i=1}^{n} w_i \left(g(\eta_i)\right) \end{pmatrix} \oplus \begin{pmatrix} (\mu_1), \\ (\xi_1), \\ (\eta_1) \end{pmatrix}$$

$= PHFWA(P_1, P_2, \dots, P_n) \oplus P_1.$

Theorem 9 (Homogeneity): The following result for a real number $\rho > 0$ holds.

$PHFWA(\rho P_1, \rho P_2, \dots, \rho P_n) = \rho PHFWA(P_1, P_2, \dots, P_n)$.

Proof: For any real digit $\rho > 0$ and for P_i. We have

$\rho P_i = \left(h^{-1}\left(h\rho(\mu_i)\right), g^{-1}\left(g\rho(\xi_i)\right), g^{-1}\left(\rho(\eta_i)\right)\right)$

Thus $PHFWA(\rho P_1, \rho P_2, \dots, \rho P_n) =$

$$\begin{pmatrix} h^{-1}\sum_{i=1}^{n} w_i\, h\left(h^{-1}\rho h(\mu_i)\right), \\ g^{-1}\sum_{i=1}^{n} w_i\, g\left(g^{-1}\rho g(\xi_i)\right), \\ g^{-1}\sum_{i=1}^{n} w_i\, g\left(g^{-1}\rho g(\eta_i)\right) \end{pmatrix}$$

$$= \begin{pmatrix} h^{-1}\sum_{i=1}^{n} w_i\left(\rho h(\mu_i)\right), \\ g^{-1}\sum_{i=1}^{n} w_i\left(\rho g(\xi_i)\right), \\ g^{-1}\sum_{i=1}^{n} w_i\left(\rho g(\eta_i)\right) \end{pmatrix}$$

$$= \begin{pmatrix} h^{-1}\left(\rho h\left(h^{-1}\left\{\sum_{i=1}^{n} w_i\left(h(\mu_i)\right)\right\}\right)\right), \\ g^{-1}\left(\rho g\left(g^{-1}\left\{\sum_{i=1}^{n} w_i\left(\rho g(\xi_i)\right)\right\}\right)\right), \\ g^{-1}\left(\rho g\left(g^{-1}\left\{\sum_{i=1}^{n} w_i\left(\rho g(\eta_i)\right)\right\}\right)\right) \end{pmatrix}$$

$\rho PHFWA\ (P_1, P_2,...,P_n)$.

If we consider the function $g(t)$ in some other form. Then we obtained some different aggregation operators e.g. if we take

$g(t) = -\text{Log}(t)$ then using Eq (1) we have

$PHFWA\ (P_1, P_2,...,P_n)$

$$= \begin{pmatrix} 1-\prod_{i=1}^{n}\left(1-(\mu_i)\right)^{w_i}, \\ \prod_{i=1}^{n}(\xi_i)^{w_i}, \prod_{i=1}^{n}(\eta_i)^{w_i} \end{pmatrix}$$

Such aggregation operator is known as PHF Archimedean weighted averaging operator.

Sometime, the ordered position of an objects is what we are interested in. So, the idea of OWA operators is used to proposed PHFOWA operators.

Definition 14: The PHFOWA operator for some P_i is defined as:

$$PHFOWA(P_1, P_2, \dots, P_n) = \oplus_{i=1}^{n} w_i P_{o(i)}$$

Where $\left(o(1), o(2),...., o(n)\right)$ are the permutations such that $Po(j\text{-}1) \geq Po(j)$ for $j = 2,3,...,n$ and $Po(j)$ is the j^{th} greatest of PHFNs Pj.

Theorem 10: The PHFOWA operator for some P_i is defined as:

$PHFOWA\ (P_1, P_2,...,P_n) =$

$$\begin{pmatrix} h^{-1}\sum_{i=1}^{n} w_i h\left(\mu_{o(i)}\right), \\ g^{-1}\sum_{i=1}^{n} w_i g\left(\xi_{o(i)}\right), \\ g^{-1}\sum_{i=1}^{n} w_i g\left(\eta_{o(i)}\right) \end{pmatrix} \rightarrow (5)$$

If $\xi_i = 0 \forall\ i$ then Eq. (5) reduces to the environment of IFSs.

$PHFOWA(P_1, P_2, \dots, P_n)$

$$= \left(h^{-1}\sum_{i=1}^{n} W_i h\left(\mu_{o(i)}\right), g^{-1}\sum_{i=1}^{n} W_i g\left(\eta_{o(i)}\right)\right)$$

Proof: Proof is straightforward.

Example 6: Let

$P_1 = (\{0.1,0.2\}, \{0.2,0.3\}, \{0.3,0.1\})$, $P_2 = (\{0.1,0.1\}, \{0.2,0.2\}, \{0.1,0.2\})$ and $P_3 = (\{0.2,0.3\}, \{0.3,0.1\}, \{0.2,0.1\})$ be any three PHFNs and $w = (0.1,0.2,0.7)^T$ be the weight vector, so $S(P_1) = -0.1$, $S(P_2) = -0.083$ and $S(P_3) = -0.033$. Since $S(P_3) > S(P_2) > S(P_1)$ so

$P_{o(1)} = P_3 =$

$(\{0.2,0.3\}, \{0.3,0.1\}, \{0.2,0.1\})$, $P_{o(2)} = P_2 = (\{0.1,0.1\}, \{0.2,0.2\}, \{0.1,0.2\})$ and

$P_{o(3)} = P_1 =$

$(\{0.1,0.2\}, \{0.2,0.3\}, \{0.3,0.1\})$. Then by Eq. (4) corresponding to positive function $g(t) = -\text{Log}(t)$, it follows that

$PHFOWA(P_1, P_2, P_3)$

$$= \begin{pmatrix} 1-\prod_{i=1}^{3}\left(1-\left(\mu_{o(i)}\right)\right)^{w_i}, \\ \prod_{i=1}^{3}\left(\xi_{o(i)}\right)^{w_i}, \prod_{i=1}^{3}\left(\eta_{o(i)}\right)^{w_i} \end{pmatrix} (6)$$

$= (\{0.12,0.20\}, \{0.16,0.21\}, \{0.20,0.08\})$

The properties of idempotency, monotonicity, boundedness, shift invariance, and homogeneity holds for PHFOWA operators obviously. Here we discussed another useful result.

Property 1: For some P_i. If

(1) $w = (1,0,...,0)^T$ then

$PHFOWA\ (P_1, P_2,...,P_n) = \max(P_1, P_2,...,P_n)$

(2) $w = (0,0,...,1)^T$ then

$PHFOWA\ (P_1, P_2,...,P_n) = \min(P_1, P_2,...,P_n)$

(3) $w_i = 1$, and $w_i = 0\ (i \neq j)$, then

$PHFOWA\ (P_1, P_2,...,P_n) = P_{o(i)}$ and $P_{o(i)}$ is the i^{th} largest of P_i.

Whenever, if one need to weight the PHFN as well as its ordered position, the hybrid aggregation operators could be of use. Therefore, we introduced hybrid operator i.e. PHFHA operators.

Definition 15: For some P_i. The PHFHA operator is defined as

$PHFHA\ (P_1, P_2,...,P_n)$

$$= \oplus_{i=1}^{n} w_i P^{\cdot}_{o(\iota)}$$

Where w_i is weight vector of PHFHA operator, $P^{\cdot}_{o(\iota)}$ is the i[th] greatest of the weighted PHFNs $\dot{P}_i\left(\dot{P}_i = n\dot{w}_i P_i, i = 1,2,\dots,n\right)$, n is the number of PHFNs and $\dot{w} = (\dot{w}_1, \dot{w}_2, \dots, \dot{w}_n)^T$ is the weight vector of P_i.

Based on operational rules of PHFSs we have the following results:

Theorem 11: For some P_i. The PHFHA operator is defined as:

$PHFHA\,(P_1, P_2, \dots, P_n)$

$$= \begin{pmatrix} h^{-1}\sum_{i=1}^{n} w_i h\left(\mu_{o(\iota)}^{\cdot}\right), \\ g^{-1}\sum_{i=1}^{n} w_i g\left(\xi_{o(\iota)}^{\cdot}\right), \\ g^{-1}\sum_{i=1}^{n} w_i g\left(\eta_{o(\iota)}^{\cdot}\right) \end{pmatrix} \rightarrow (7)$$

Proof: Proof is straightforward.

The properties of idempotency, monotonicity, boundedness, shift invariance and homogeneity holds true for PHFHA operators.

Theorem 12: If $w = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)^T$ then PHFHA operator becomes PHFWA operator.

Proof: As $P_{o(\iota)}^{\cdot} = n\dot{w}_i P_i$ and $w = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)^T$ thus $w_i P_{o(\iota)}^{\cdot} = \dot{w}_i P_i$ and hence

$PHFHA\,(P_1, P_2, \dots, P_n)$

$$= w_1 P_{o(1)}^{\cdot} \oplus w_2 P_{o(2)}^{\cdot} \oplus \dots\dots \oplus w_n P_{o(n)}^{\cdot}$$

$$= \dot{w}_1 P_1 \oplus \dot{w}_2 P_2 \oplus \dots\dots \oplus \dot{w}_n P_n$$

$$= PHFWA(P_1, P_2, \dots, P_n).$$

Theorem 13: If $w = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)^T$ then, PHFHA operator becomes PHFOWA operator.

Proof: Proof is similar to theorem 12.

V. APPLICATION

In this section, we used the aggregation operators defined in section four in a DM problem in the environment of PHFSs.

In DM problems we usually have *m* alternatives to be evaluated keeping *m* attributes/criterion under observation. In current situation, the DM has to be took place in the environment of PHFSs. Let us assume the alternatives x_j(j = 1,2,..,m) based on the criterion G_i(i = 1,2,..,m) need to evaluate by the decision maker in the form of PHFSs. Then for finding suitable alternative we need to follow the following steps.

Step 1: Construction the decision matrix.

$$\begin{array}{c} \\ A1 \\ A2 \\ \dots \\ \dots \\ Am \\ \\ \end{array} \begin{pmatrix} G1 & G2 & \dots & Gn \\ ((\mu_{11}),(\xi_{11}),(\eta_{11})) & ((\mu_{12}),(\xi_{12}),(\eta_{12})) & \dots & ((\mu_{1n}),(\xi_{1n}),(\eta_{1n})) \\ ((\mu_{21}),(\xi_{21}),(\eta_{21})) & ((\mu_{22}),(\xi_{22}),(\eta_{22})) & \dots & ((\mu_{2n}),(\xi_{2n}),(\eta_{2n})) \\ \dots\dots & \dots\dots\ \dots\dots & & \dots\dots\dots\ \ \dots\dots \\ \dots\dots & \dots\dots\dots\dots & & \dots\dots\dots\ \ \dots\dots \\ \dots\dots & \dots\dots\dots\dots\dots & & \dots\dots\dots\ \ \dots. \\ ((\mu_{m1}),(\xi_{m1}),(\eta_{m1})) & ((\mu_{m2}),(\xi_{m2}),(\eta_{m2})) & \dots & ((\mu_{mn}),(\xi_{mn}),(\eta_{mn})) \end{pmatrix}$$

Step 2: (Decision matrix is normalized). If the criterion is cost or benefit type then normalization of decision matrix takes place according to $r_{ij} = \begin{Bmatrix} P_{ij}{}^{C} & k \in B \\ P_{ij} & k \in C \end{Bmatrix}$ →(8) where $P_{ij}{}^{C}$ is the compliment of P_{ij}.

Step 3: The information provided by decision makers is aggregated.

Step 4: Computing the score values of date obtained in step 3.

Step 5: ordering of alternatives.

The following example adapted from [xxi] with slight modification is established in support of MADM process discussed above.

Example 8: A financial strategy is to be planned by a firm for the upcoming year. Some initial screening has been done and some survey reports have been prepared after which the company has three strategies that needs to be evaluated. These strategies include:

- A_1: as to investment in the Asian markets.
- A_2: as to investment in the British markets.
- A_3: as to investment in the European markets.

This evaluation proceeds from the following aspects as

- G_1: the analysis of the growth.
- G_2: the analysis of the risk.
- G_3: the analysis of social and political impact.

We are taking the weight vector is $w = (0.3,0.2,0.5)^T$ To get the best strategy among the three available, we use PHFWA operator. The evaluation based on PHFOWA and PHFHA operators could be done analogously. The step by step calculations are as follows:

Step 1: Information of decision makers about alternative $A_i(i = 1,2,3)$ in decision matrix i.e. formation of decision matrix

$$\begin{pmatrix} (\{0.1,0.2\},\{0.2,0\},\{0.1,0.2\}) & (\{0.2,0.1\},\{0.3,0.2\},\{0.4,0.4\}) & (\{0.2,0.1\},\{0.1,0.2\},\{0.2,0.3\}) \\ (\{0.1,0.0\},\{0.1,0.1\},\{0.2,0.1\}) & (\{0.3,0.2\},\{0.2,0.2\},\{0.1,0.2\}) & (\{0.4,0.5\},\{0.1,0.2\},\{0.1,0.1\}) \\ (\{0.2,0.1\},\{0.1,0.2\},\{0.2,0.2\}) & (\{0.1,0.1\},\{0.2,0.1\},\{0.1,0.2\}) & (\{0.1,0.2\},\{0.1,0.1\},\{0.2,0.1\}) \end{pmatrix}$$

Step 2: Here the criteria G_2 and G_3 are the cost criteria while G_1 is the benefit criteria so we need to normalize the matrix.

$R=(r_{ij})_{3x3}$ Using Eq. (8) as follows

$$R = \begin{bmatrix} (\{0.1,0.2\},\{0.2,0\},\{0.1,0.2\}) & (\{0.4,0.4\},\{0.3,0.2\},\{0.2,0.1\}) & (\{0.2,0.3\},\{0.1,0.2\},\{0.2,0.1\}) \\ (\{0.2,0.1\},\{0.1,0.1\},\{0.1,0\}) & (\{0.1,0.2\},\{0.2,0.2\},\{0.3,0.2\}) & (\{0.1,0.1\},\{0.1,0.2\},\{0.4,0.5\}) \\ (\{0.2,0.2\},\{0.1,0.2\},\{0.2,0.1\}) & (\{0.1,0.2\},\{0.2,0.1\}\{0.1,0.1\}) & (\{0.2,0.1\},\{0.1,0.1\},\{0.1,0.2\}) \end{bmatrix}$$

Step 3: By following the PHFWA the generator are follows $g(t) = -\text{Log}(t)$, we obtained the all of aggregated value of each alternative A_i as

$R_1 = PHFWA(r_{11},r_{12},r_{13})$
$= (\{0.22,0.28\}, \{0.15,0\}, \{016,0.12\})$

$R_2 = PHFWA(r_{21},r_{22},r_{23})$
$= (\{0.12,0.11\}, \{0.11,0.16\}, \{0.24,0.0\})$

$R_3 = PHFWA(r_{21},r_{22},r_{23})$
$= (\{0.18,0.37\}, \{0.11,0.12\}, \{0.12,0.14\})$

Step 4: The aggregated data of $R_i(i = 1,2,3)$ are using the definitions of the score function such that $S(R_1) = 0.0116 S(R_2) = -0.046$ and $S(R_3) = 0.01$

Step 5: Since $S(R_1) > S(R_3) > S(R_2)$ thus we have $A_1 > A_3 > A_2$. Hence, the best financial strategy is A_1 i.e. to invest in the Asian market.

Hence using the aggregation operators of PHFSs, the selection of best financial policy has been carried out. The importance of working in the environment of PHFSs lie in the fact that this kind of framework enables us to handle the information of PFSs, HFSs, IFSs, IHFS etc. On the other hand, the frameworks of IFSs, PFSs, HFSs, IHFSs are unable to process the information of PHFSs. All this shows the superiority of PHFSs.

VI. CONCULSION

In this manuscript, the concept of PHFSs is introduced as a combination of PFSs and HFSs. We investigated the properties of thebasic results of PHFSs and explained them numerically. Based on defined operations, some aggregation operators for proposed PHFSs have been developed including PHFWA, PHFOWA and PHFHA operators. The different properties of defined aggregation operators have been discussed. We also discussed the MADM process in the environment of PHFSs and in support of our proposed method a numerical example is discussed. The idea of PHFSs is a generalization of PFSs and could be very useful in real life applications. In future, some similarity measures can be established for PHFSs and can be applied in practical problems.

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