Technical Journal, University of Engineering and Technology (UET) Taxila, Pakistan Vol. 24 No. 1-2019
ISSN:1813-1786 (Print) 2313-7770 (Online)

Analysis of Frechet Distribution Based on Type-I Censored Data

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Abstract- In this paper an effort has been made to develop maximum likelihood estimators of two parameters Frechet distribution based on type-I censored data. The observed Fisher information matrix and confidence intervals of the parameters based on asymptotic normality are also derived. An extensive simulation study is conducted to illustrate the statistical properties of the parameters and the confidence intervals. Finally, one real data set about the remission times of leukemia patients is analyzed for illustrative purposes.

Keywords- Frechet distribution, asymptotic variance, Maximum Likelihood Estimation, Confidence Interval, Asymptotic Confidence Interval.

I. INTRODUCTION

The problem of censoring is more commonly encountered in lifetime tests and reliability analysis because no experiment may remain sustained for an infinite time due to constraints on the available time or cost for testing. There are different types of censoring schemes which include right, left, interval, hybrid, random, type-I and type-II censoring. The two most common types of censoring are type-I censoring (time censoring) and type-II censoring (failure censoring). In type-I censoring, the test runs for a preselected time and in type-II censoring, the test stops at the occurrence of preselected number of failures. In this study, we consider type-I censored lifetime data, and assume that lifetime of the experimental unit follows the Frechet distribution (FD).

FD was introduced by French mathematician Maurice Frechet (1878-1973) [1], who identified possible limit distribution for the largest order statistic during 1927. The FD is a special case of the generalized extreme value distribution. The FD has been used as an useful method for analyzing several extreme events and linked to modeling of several real-world phenomenon such as human lifetimes, accelerated life testing, earthquake, flood, rainfall, radioactive emissions, seismic analysis, sea current and wind speed [2-4]. Therefore, FD is well suited to characterize random variables of large components. Cumulative distribution function of FD is;

$$F(t) = exp\left[-\left(\frac{\beta}{t}\right)^{\alpha}\right] \quad (1)$$

Therefore, corresponding probability density function of the FD is;

$$f(t;\alpha,\beta) = \frac{\alpha}{\beta}\left(\frac{\beta}{t}\right)^{\alpha+1} exp\left[-\left(\frac{\beta}{t}\right)^{\alpha}\right] \quad (2)$$

This distribution has a heavy upper tail and is bounded on the lower side ($t > 0$). The survival function is

$$S(t) = 1 - exp\left[-\left(\frac{\beta}{t}\right)^{\alpha}\right]$$

Here the parameter α determines the shape of the distribution and β is the scale parameter. FD is equivalent to taking the reciprocal of values from a standard Weibull distribution. FD is often called the Frechet model in engineering circles. Extensive work has been done on the FD since then, both from the frequentist and Bayesian points of view [5-8]. Book length work has been done on two-parameters FD [9-13]. Recently, [14] has proposed an optimization of simple step stress accelerated life test for the FD under type-I censoring. In the present study, we have proposed maximum likelihood estimators (MLE) for FD based on type-I censored data and comparison among different estimates are made in terms of variances using different sample size and different arbitrary terminating time. Including this introduction section, the rest of the paper is arranged as follows. Section 2, comprises the derivation of MLEs. Fisher Information matrix and asymptotic confidence interval (CI) are discussed in Section 3. Simulation study is carried out in Section 4 and a real data set is analyzed and results presented in Section 5 for illustrative purposes. Finally, conclusion is given in Section 6.

II. MAXIMUM LIKELIHOOD ESTIMATION

Let T_1, T_2, …,T_n is a type-I censored sample of 'n' items whose lifetimes have the FD with predetermined time 't_c'. The likelihood function for type-I censored values is

$$l(\alpha,\beta|t) = \left[1 - exp\left[-\left(\frac{\beta}{t_c}\right)^{\alpha}\right]\right]^{n-d} \prod_{i=1}^{n}\left[\frac{\alpha}{\beta}\left(\frac{\beta}{t_i}\right)^{\alpha+1} exp\left[-\left(\frac{\beta}{t_i}\right)^{\alpha}\right]\right]^{\delta_i} \quad (3)$$

Where $d = \sum_{i=1}^{n}\delta_i$ is a random variable and $\delta_i = I(t_i < t_c)$ is an indicator function. i.e.,

$$\delta_i = \begin{cases} 1 & t_i < t_c \\ 0 & t_i \geq t_c \end{cases}$$

In practice, it is often more convenient to work with the logarithm of the likelihood function, called the log-likelihood. The log-likelihood of (3) is

$$l(\alpha,\beta|t) = d\ln\alpha + d\alpha\ln\beta - (\alpha+1)\sum_{i=1}^{n}\delta_i \ln t_{(i)} - \sum_{i=1}^{n}\delta_i\left(\frac{\beta}{t_i}\right)^{\alpha} + (n-d)\ln\left[1 - exp\left[-\left(\frac{\beta}{t_i}\right)^{\alpha}\right]\right]$$

The MLE's of α and β say α^ and β^can be obtained as the solutions of the following score equations

$$\frac{\partial l}{\partial\alpha} = \frac{d}{\alpha} + d\ln\beta - \sum_{i=1}^{n}\delta_i \ln t_{(i)} - \sum_{i=1}^{n}\delta_i\left(\frac{\beta}{t_{(i)}}\right)^{\alpha}\ln\left(\frac{\beta}{t_{(i)}}\right)^{\alpha} + \left[\frac{(n-d)\left(\frac{\beta}{t_c}\right)^{\alpha}\ln\left(\frac{\beta}{t_c}\right)exp\left[-\left(\frac{\beta}{t_c}\right)^{\alpha}\right]}{1 - exp\left[-\left(\frac{\beta}{t_c}\right)^{\alpha}\right]}\right] = 0, (4)$$

$$\frac{\partial l}{\partial\beta} = \frac{d\alpha}{\beta} - \frac{\alpha}{\beta}\sum_{i=1}^{n}\delta_i\left(\frac{\beta}{t_{(i)}}\right)^{\alpha} + \left[\frac{(n-d)\left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{t_c}\right)^{\alpha}exp\left[-\left(\frac{\beta}{t_c}\right)^{\alpha}\right]}{1-exp\left[-\left(\frac{\beta}{t_c}\right)^{\alpha}\right]}\right] = 0, (5)$$

Apparently, above equations cannot be written in closed form so the MLEs can be obtained numerically. So, an iterative procedure can be used to get ML estimates. Here we use the BFGS Quasi Newton Optimization method which is available in the R software. Further, for interval estimates of model parameters, we need Fisher information matrix.

III. INTERVAL ESTIMATES

To construct the CIs of α and β an approximate method is used to provide good probability for large sample sizes and is easy to compute. Elements of Fisher information matrix (FIM) are obtained numerically. After that, the asymptotic normality of MLEs is used to construct approximate CIs for α and β. The FIM is derived by taking the negative second partial derivatives of log-likelihood function with respect to α and β. The asymptotic FIM is:

$$I_{(\hat{\alpha},\hat{\beta})} = \begin{pmatrix} -\frac{\partial^2 l}{\partial\alpha^2} & -\frac{\partial^2 l}{\partial\alpha\partial\beta} \\ -\frac{\partial^2 l}{\partial\alpha\partial\beta} & -\frac{\partial^2 l}{\partial\beta^2} \end{pmatrix}$$

$$\frac{\partial^2 l}{\partial\alpha^2} = -\frac{d}{\alpha^2} - \sum_{i=1}^{n}\delta_i\left(\frac{\beta}{t_i}\right)^{\alpha}\ln\left(\frac{\beta}{t_i}\right)^{2} - p\left[\left(\frac{\beta}{t_c}\right)^{\alpha} - 1 + \frac{\left(\frac{\beta}{t_c}\right)^{\alpha}exp\left[-\left(\frac{\beta}{t_c}\right)^{\alpha}\right]}{1 - exp\left[-\left(\frac{\beta}{t_c}\right)^{\alpha}\right]}\right]$$

$$\frac{\partial^2 l}{\partial\beta^2} = \frac{\alpha}{\beta^2}\left[-d - (\alpha-1)\sum_{i=1}^{n}\delta_i\left(\frac{\beta}{t_i}\right)^{\alpha} + (\alpha-1)Q - \alpha\beta^{\alpha}Q - (Q)\frac{\alpha\beta^{\alpha}exp\left[-\left(\frac{\beta}{t_c}\right)^{\alpha}\right]}{1 - exp\left[-\left(\frac{\beta}{t_c}\right)^{\alpha}\right]}\right]$$

$$\frac{\partial^2 l}{\partial\alpha\partial\beta} = \frac{d}{\beta} - \frac{\alpha}{\beta}\sum_{i=1}^{n}\delta_i\left(\frac{\beta}{t_i}\right)^{\alpha}\ln\left(\frac{\beta}{t_i}\right) - \frac{1}{\beta}\sum_{i=1}^{n}\delta_i\left(\frac{\beta}{t_i}\right)^{\alpha} - Q\left(\frac{\alpha}{t_c}\right)\left(\frac{\beta}{t_c}\right)^{\alpha-1}\ln\left(\frac{\beta}{t_c}\right) - Q\left(\frac{\alpha}{\beta}\right)\ln\left(\frac{\beta}{t_c}\right) - \left(\frac{Q}{\beta}\right) + Q\left[\frac{\left(\frac{\beta}{t_c}\right)^{\alpha-1}\ln\left(\frac{\beta}{t_c}\right)exp\left[-\left(\frac{\beta}{t_c}\right)^{\alpha}\right]\left(\frac{1}{t_c}\right)}{1 - exp\left[-\left(\frac{\beta}{t_c}\right)^{\alpha}\right]}\right]$$

Where $P = (n-d)\left[\ln\left(\frac{\beta}{t_c}\right)\right]^2 exp\left[-\left(\frac{\beta}{t_c}\right)^{\alpha}\right]\left(\frac{\beta}{t_c}\right)^{\alpha}\left[1 -\right.$

$$exp\left[-\left(\frac{\beta}{t_c}\right)^{\alpha}\right]\Bigg]^{-1},$$

$$Q = (n-d)exp\left[-\left(\frac{\beta}{t_c}\right)^{\alpha}\right]\left(\frac{\beta}{t_c}\right)^{\alpha}\left[1 - exp\left[-\left(\frac{\beta}{t_c}\right)^{\alpha}\right]\right]^{-1}$$

Moreover, the CI with an approximate confidence coefficient (1 - γ) for the MLEs $\hat{\theta}$ of a parameter θ. i.e., $\hat{\alpha}$ and $\hat{\beta}$ can be obtained as:

$$P\left[\hat{\theta} - z(\hat{\theta}) \leq \theta \leq \hat{\theta} + z\sigma(\hat{\theta})\right] \approx 1-\gamma$$

where 'z' is the $\left[100\left(\frac{1-\gamma}{2}\right)\right]$th standard normal percentile. The two sided approximate CI for α and β can be obtained as:

$$\hat{\alpha} \pm z\left(\frac{1-\gamma}{2}\right)\sqrt{\hat{\sigma}^2}(\hat{\alpha}), \hat{\beta} \pm z\left(\frac{1-\gamma}{2}\right)\sqrt{\hat{\sigma}^2}(\hat{\beta}) \quad (6)$$

The asymptotic variance of the estimated parameters α and β are obtained by asymptotic variance covariance matrix which is designed as the inverse of the FIM. Thus, the two sided approximate CIs for parameters α and β under different sample sizes will be constructed using equation (6) with confidence levels 90% and 95%.

IV. Simulation Study

To evaluate the performance of proposed MLEs, samples are generated from the FD using inverse transformation and we replicated the process 1000 times. In this section, ML estimates are compared in terms of variance regarding various sample sizes and various parametric values using predetermined terminating time "t_c' taken as 3, 5, 8 and 10. Sample size is varied to see the behavior of small and large samples and sample size is taken as n = 30, 50, 75, 100, 150 and 200. Based on the asymptotic normality, the approximate 90% and 95% CIs of the parameters are obtained. The values of α and β are taken as (2, 1.5), (3, 1.5), (2, 2) respectively. Tables I-XII summarize the results of the point and interval estimates for α and β. From the results, conclusions are drawn regarding the behavior of the estimators, which are summarized as follows:

From Tables I-XII, it is observed that as the sample size increases the variance of the estimates get smaller. It can be visualized from figures 1-6.

It can also be observed from Table I-XII that the values of estimates are closer to the true values at terminating time t_c = 10, as compared to other terminating time.

TABLE I.
AVERAGE POINT AND INTERVAL ESTIMATES FOR α = 2, β = 1.5, AND t_c = 3.

n	Par.	ML	Var.	90 % CI	95% CI
30	α	3.00	0.2266	2.23, 3.77	2.09, 3.92
	β	1.28	0.0096	1.12, 1.44	1.09, 1.47
50	α	2.91	0.1236	2.34, 3.48	2.23, 3.59
	β	1.28	0.0059	1.15, 1.40	1.13, 1.42
75	α	2.87	0.0784	2.41, 3.33	2.32, 3.41
	β	1.27	0.0039	1.17, 1.38	1.15, 1.40
100	α	2.85	0.0579	2.46, 3.24	2.38, 3.32
	β	1.27	0.0029	1.18, 1.36	1.17, 1.38
150	α	2.82	0.0373	2.51, 3.14	2.45, 3.20
	β	1.27	0.0019	1.20, 1.34	1.18, 1.36
200	α	2.81	0.0276	2.54, 3.09	2.49, 3.14
	β	1.27	0.0015	1.21, 1.34	1.20, 1.35

Par. Parameter; Var. Variance

TABLE II
AVERAGE POINT AND INTERVAL ESTIMATES FOR *α* = 2, *β* = 1.5, AND t_c = 5.

n	Par.	ML	Var.	90 % CI	95% CI
30	α	2.49	0.1362	1.89, 3.09	1.78, 3.20
	β	1.40	0.0138	1.21, 1.59	1.17, 1.63
50	α	2.43	0.0762	1.98, 2.88	1.90, 2.97
	β	1.39	0.0084	1.25, 1.54	1.22, 1.57
75	α	2.42	0.0533	2.04, 2.79	1.97, 2.87
	β	1.39	0.0060	1.27, 1.52	1.24, 1.54
100	α	2.40	0.0365	2.09, 2.71	2.03, 2.77
	β	1.39	0.0042	1.28, 1.50	1.26, 1.52
150	α	2.38	0.0238	2.13, 2.64	2.08, 2.69
	β	1.39	0.0028	1.30, 1.48	1.29, 1.49
200	α	2.38	0.0177	2.16, 2.59	2.12, 2.64
	β	1.39	0.0021	1.31, 1.47	1.30, 1.48

TABLE III
AVERAGE POINT AND INTERVAL ESTIMATES α = 2, β = 1.5, AND t_c = 8.

n	Par.	ML	Var.	90 % CI	95% CI
30	α	2.28	0.1106	1.74, 2.82	1.64, 2.93
	β	1.46	0.0170	1.25, 1.67	1.21, 1.71
50	α	2.23	0.0623	1.82, 2.64	1.75, 2.72
	β	1.45	0.0102	1.29, 1.62	1.26, 1.65
75	α	2.21	0.0405	1.88, 2.54	1.82, 2.61
	β	1.45	0.0068	1.31, 1.58	1.29, 1.61
100	α	2.20	0.0301	1.92, 2.49	1.87, 2.54
	β	1.45	0.0051	1.33, 1.56	1.31, 1.59
150	α	2.19	0.0196	1.96, 2.42	1.92, 2.46
	β	1.45	0.0034	1.35, 1.54	1.33, 1.56
200	α	2.18	0.0146	1.98, 2.38	1.95, 2.42
	β	1.45	0.0026	1.36, 1.53	1.35, 1.55

ISSN:1813-1786 (Print) 2313-7770 (Online)

TABLE IV
AVERAGE POINT AND INTERVAL ESTIMATES FOR $\alpha = 2$, $\beta = 1.5$, AND $t_c = 10$.

n	Par.	ML	Var.	90 % CI	95% CI
30	α	2.22	0.1044	1.70, 2.75	1.60, 2.85
	β	1.47	0.0181	1.26, 1.67	1.22, 1.73
50	α	2.17	0.0589	1.78, 2.57	1.70, 2.65
	β	1.47	0.0109	1.30, 1.64	1.27, 1.67
75	α	2.16	0.0383	1.84, 2.48	1.77, 2.54
	β	1.46	0.0072	1.33, 1.60	1.30, 1.63
100	α	2.15	0.0285	1.87, 2.43	1.82, 2.48
	β	1.46	0.0054	1.34, 1.58	1.13, 1.61
150	α	2.14	0.0186	1.91, 2.36	1.87, 2.40
	β	1.46	0.0036	1.36, 1.56	1.34, 1.58
200	α	2.13	0.0139	1.94, 2.32	1.90, 2.36
	β	1.40	0.0027	1.38, 1.55	1.36, 1.57

A. Graphical Representation of variances when $\alpha = 2$ and $\beta = 1.5$

Further graphs of variances are constructed to check the behavior of variances of estimates. We used Microsoft Excel for the construction of graphs. Considering the values of $\alpha = 2$ and $\beta = 1.5$.

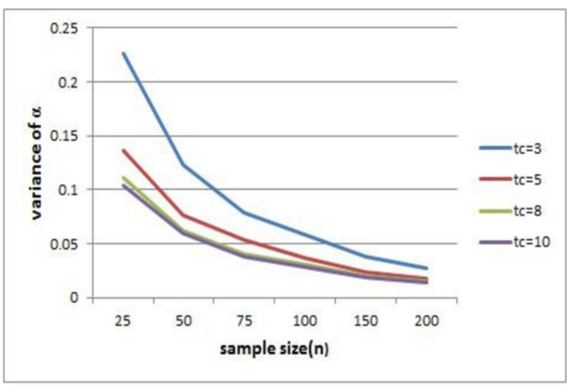

Fig.1. Graph of variance of $\acute{\alpha}$ when $\alpha = 2$ and $\beta = 1.5$.

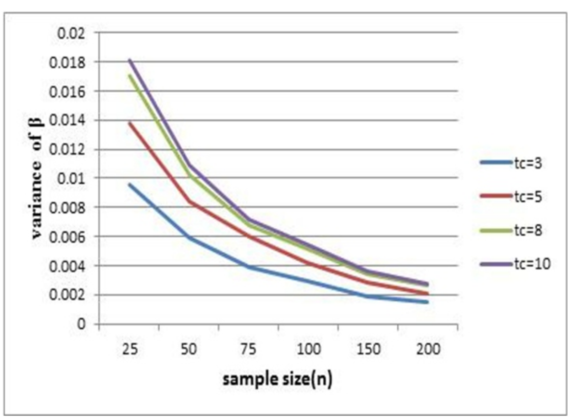

Fig. 2. Graph of variance of $\hat{\beta}$ when $\alpha = 2$ and $\beta=1.5$

It can be observed from results in TABLE I-IV that when the sample size increases, the interval of the estimates decreases. Fig. 1 and 2, clearly indicates the decrement in variance with increase in sample size.

TABLE V
AVERAGE POINT AND INTERVAL ESTIMATES FOR $\alpha = 3$, $\beta = 1.5$, and $t_c = 3$.

n	Par.	ML	Var.	90% CI	95% CI
30	α	3.93	0.3472	2.97, 4.88	2.79, 5.06
	β	1.41	0.0059	1.28, 1.53	1.26, 1.56
50	α	3.82	0.1924	3.11, 4.52	2.97, 4.67
	β	1.41	0.0036	1.31, 1.50	1.29, 1.52
75	α	3.78	0.1240	3.21, 4.36	3.10, 4.47
	β	1.40	0.0024	1.32, 1.48	1.31, 1.50
100	α	3.77	0.0915	3.27, 4.26	3.18, 4.36
	β	1.40	0.0018	1.33, 1.47	1.32, 1.49
150	α	3.74	0.0597	3.34, 4.14	3.26, 4.22
	β	1.40	0.0012	1.35, 1.46	1.33, 1.47
200	α	3.73	0.0443	3.38, 4.07	3.31, 4.14
	β	1.40	0.0009	1.35, 1.45	1.34, 1.46

TABLE VI
AVERAGE POINT AND INTERVAL ESTIMATES FOR $\alpha = 3$, $\beta = 1.5$, AND $t_c = 5$.

n	Par.	ML	Var.	90% CI	95% CI
30	α	3.37	0.2401	2.57, 4.17	2.42, 4.32
	β	1.48	0.0077	1.33, 1.62	1.30, 1.65
50	α	3.29	0.1353	2.69, 3.90	2.58, 4.01
	β	1.47	0.0048	1.36, 1.59	1.34, 1.61
75	α	3.27	0.0879	2.78, 3.75	2.69, 3.85
	β	1.47	0.0032	1.38, 1.56	1.36, 1.58
100	α	3.26	0.0653	2.84, 3.68	2.76, 3.76
	β	1.47	0.0023	1.39, 1.55	1.37, 1.56
150	α	3.24	0.0428	2.90, 3.57	2.83, 3.64
	β	1.47	0.0016	1.40, 1.53	1.39, 1.55
200	α	3.22	0.0318	2.93, 3.52	2.88, 3.57
	β	1.47	0.0012	1.41, 1.53	1.40, 1.54

TABLE VII
AVERAGE POINT AND INTERVAL ESTIMATES FOR $\alpha = 3$, $\beta = 1.5$, and $t_c = 8$.

n	Par.	ML	Var.	90% CI	95% CI
30	α	3.22	0.2183	2.46, 3.98	2.32, 4.13
	β	1.49	0.0088	1.34, 1.64	1.31, 1.67
50	α	3.15	0.1231	2.57, 3.72	2.46, 3.83
	β	1.49	0.0053	1.37, 1.61	1.35, 1.63
75	α	3.12	0.0801	2.65, 3.58	2.56, 3.67
	β	1.49	0.0035	1.39, 1.59	1.37, 1.60
100	α	3.11	0.0594	2.72, 3.51	2.63, 3.58
	β	1.49	0.0026	1.40, 1.57	1.39, 1.59
150	α	3.09	0.0389	2.76, 3.41	2.70, 3.47
	β	1.49	0.0017	1.42, 1.56	1.41, 1.57
200	α	3.08	0.0289	2.80, 3.36	2.74, 3.41
	β	1.49	0.0013	1.43, 1.55	1.42, 1.56

TABLE VIII
AVERAGE POINT AND INTERVAL ESTIMATES FOR $\alpha = 3$, $\beta = 1.5$, and $t_c = 10$.

n	Par.	ML	Var.	90% CI	95% CI
30	α	3.19	0.2144	2.44, 3.94	2.29, 4.09
	β	1.50	0.0089	1.34, 1.65	1.32, 1.68
50	α	3.12	0.1208	2.55, 3.68	2.44, 3.79
	β	1.49	0.0056	1.38, 1.64	1.35, 1.64
75	α	3.09	0.0786	2.63, 3.55	2.54, 3.63
	β	1.49	0.0038	1.40, 1.59	1.38, 1.61
100	α	3.08	0.0583	2.68, 3.47	2.61, 3.55
	β	1.49	0.0026	1.41, 1.58	1.39, 1.59
150	α	3.06	0.0382	2.74, 3.38	2.68, 3.44
	β	1.49	0.0019	1.42, 1.56	1.41, 1.57
200	α	3.05	0.0284	2.77, 3.32	2.72, 3.38
	β	1.49	0.0018	1.43, 1.55	1.42, 1.56

B. Graphical Representation of variances when $\alpha = 3$ and $\beta = 1.5$

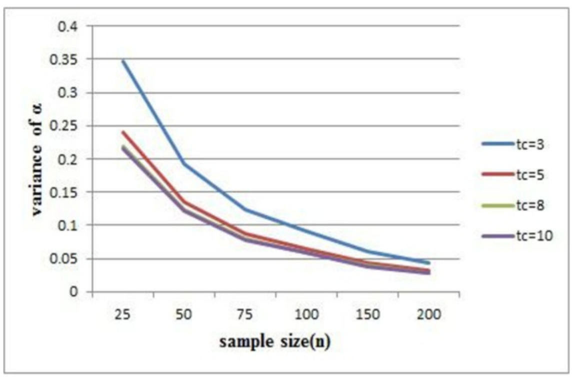

Fig. 3. Graph of variance of $\hat{\alpha}$ when $\alpha = 3$ and $\beta = 1.5$.

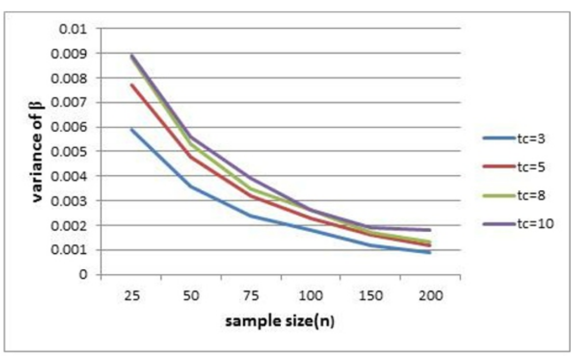

Fig. 4. Graph of variance of $\hat{\beta}$ when $\alpha = 3$ and $\beta = 1.5$.

From Fig.3 and 4 it is clearly observed that the variance of $\hat{\alpha}$ and $\hat{\beta}$ decreases as we increase sample size at all level of t_c.

TABLE IX
AVERAGE POINT AND INTERVAL ESTIMATES FOR $\alpha = 2$, $\beta = 2$, AND $t_c = 3$.

n	Par.	ML	Var.	90% CI	95% CI
30	α	3.56	0.3895	2.56, 4.56	2.76, 4.75
	β	1.58	0.0128	1.39, 1.76	1.36, 1.80
50	α	3.42	0.2064	2.69, 4.16	2.55, 4.30
	β	1.57	0.0079	1.43, 1.72	1.40, 1.74
75	α	3.35	0.1272	2.77, 3.93	2.66, 4.04
	β	1.57	0.0053	1.45, 1.69	1.42, 1.71
100	α	3.32	0.0921	2.82, 3.81	2.73, 3.91
	β	1.57	0.0040	1.46, 1.67	1.44, 1.69
150	α	3.28	0.0591	2.88, 3.68	2.81, 3.75
	β	1.56	0.0027	1.48, 1.65	1.46, 1.67
200	α	3.26	0.0435	2.92, 3.60	2.85, 3.67
	β	1.45	0.0021	1.49, 1.64	1.48, 1.65

TABLE X
AVERAGE POINT AND INTERVAL ESTIMATES FOR $\alpha = 2$, $\beta = 2$, AND $t_c = 5$.

n	Par.	ML	Var.	90% CI	95% CI
30	α	2.73	0.1721	2.05, 3.40	1.93, 3.53
	β	1.79	0.0021	1.56, 2.02	1.51, 2.07
50	α	2.65	0.0951	2.15, 3.15	2.05, 3.25
	β	1.78	0.0124	1.60, 1.97	1.57, 2.00
75	α	2.62	0.0611	2.22, 3.02	2.14, 3.10
	β	1.78	0.0083	1.63, 1.93	1.60, 1.96
100	α	2.61	0.0449	2.26, 2.96	2.19, 3.02
	β	1.78	0.0062	1.65, 1.91	1.62, 1.93
150	α	2.59	0.0293	2.31, 2.87	2.25, 2.92
	β	1.78	0.0042	1.67, 1.88	1.65, 1.90
200	α	2.58	0.0217	2.34, 2.82	2.29, 2.87
	β	1.78	0.0032	1.69, 1.87	1.67, 1.89

TABLE XI
AVERAGE POINT AND INTERVAL ESTIMATES FOR $\alpha = 2$, $\beta = 2$, AND $t_c = 8$.

n	Par.	ML	Var.	90% CI	95% CI
30	α	2.39	0.1232	1.82, 2.96	171, 3.07
	β	1.91	0.0271	1.64, 2.17	1.59, 2.22
50	α	2.34	0.0692	1.91, 2.77	1.82, 2.85
	β	1.90	0.0163	1.69, 2.11	1.65, 2.15
75	α	2.32	0.0449	1.97, 2.66	1.90, 2.73
	β	1.89	0.0109	1.72, 2.06	1.69, 2.10
100	α	2.31	0.0332	2.01, 2.61	1.95, 2.66
	β	1.89	0.0082	1.74, 2.04	1.71, 2.07
150	α	2.29	0.0217	2.05, 2.53	2.00, 2.58
	β	1.89	0.0054	1.77, 2.01	1.74, 2.03
200	α	2.28	0.0162	2.08, 2.49	2.04, 2.53
	β	1.89	0.0041	1.78, 2.00	1.76, 2.02

TABLE XII
AVERAGE POINT AND INTERVAL ESTIMATES FOR $\alpha = 2$, $\beta = 2$, AND $t_c = 10$.

n	Par.	ML	Var.	90% CI	95% CI
30	α	2.30	0.1128	1.76, 2.85	1.65, 2.95
	β	1.94	0.0296	1.66, 2.22	1.61, 2.27
50	α	2.25	0.0756	1.84, 2.66	1.76, 2.74
	β	1.93	0.0177	1.71, 2.15	1.67, 2.19
75	α	2.23	0.0413	1.90, 2.57	1.84, 2.63
	β	1.92	0.0118	1.75, 2.10	1.71, 2.14
100	α	2.22	0.0305	1.94, 2.51	1.88, 2.57
	β	1.92	0.0088	1.77, 2.08	1.74, 2.11
150	α	2.21	0.0200	1.98, 2.44	1.93, 2.49
	β	1.92	0.0059	1.80, 2.05	1.77, 2.07
200	α	2.20	0.0148	2.00, 2.40	1.96, 2.44
	β	1.92	0.0044	1.81, 2.03	1.79, 2.05

C. Graphical Representation of variances when $\alpha = 2$ and $\beta = 2$.

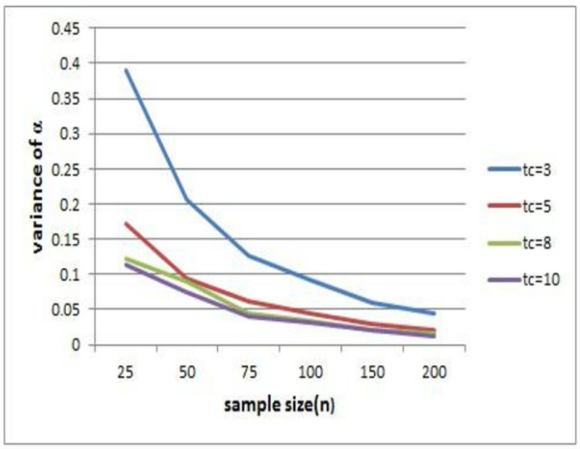

Fig.5. Graph of variance of $\hat{\alpha}$ when $\alpha = 2$ and $\beta = 2$.

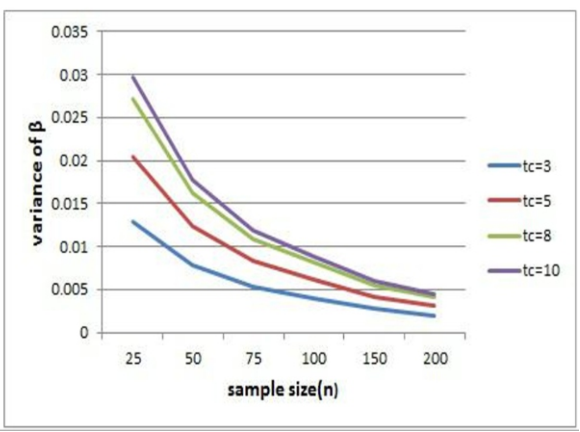

Fig. 6. Graph of variance of $\hat{\beta}$ when $\alpha = 2$ and $\beta = 2$.

V. REAL DATA ANALYSIS

To illustrate our proposed estimators, we consider the real data set which is taken from [15] about the remission times of leukemia patients due to administering a new drug. 40 patients were administered the new drug which induces the remission in leukaemia and the experiment was terminated after 210 days. The data is presented in Table XIII.

TABLE XIII.
REMISSION TIMES OF LEUKEMIA PATIENTS

45	56	58	64	77	79
89	128	131	142	144	149
163	166	175	176	184	184
188	190	191	204		

Clearly, it is a type-I censored sample with n = 40 and n-d =22. Just for computational ease, we have divided all the data points by 100. It does not affect in any inferential procedure.

TABLE XIV
Average point and interval estimates for $\alpha = 2$, $\beta = 1$, and $t_c = 18$.

n	Par.	ML	Var.	90%CI	95%CI
40	α	2.0397	0.118	1.5148 2.5646	1.4143, 2.6651
	β	0.9659	0.0115	0.7891 1.1426	0.7553, 1.1765

As shown from Table XIV, the width of CIs is too narrow which indicates that point estimates are stable for this data set. Now the natural question is whether FD provides a good fit or not. We compute the Kolmogorov-Smirnov distances between the empirical distribution function and the fitted distribution function based on MLEs, and it is D=0.25126 and the associated p-value is 0.1243. Therefore, based on p-value, we can say that FD fits quite well to the above data set. Further, Fig. 7. shows the empirical and fitted CDFs using MLE.

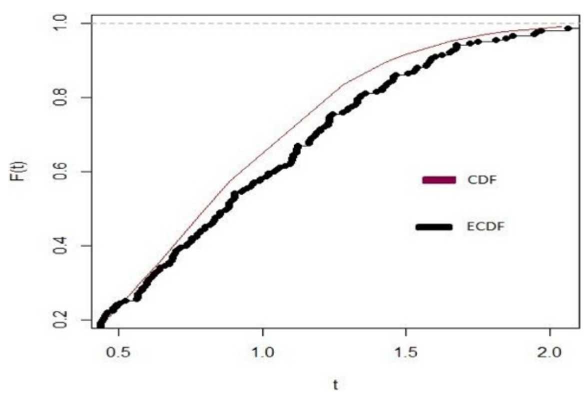

Fig. 7. Empirical and fitted CDF using MLE.

VI. Conclusion

In this paper we consider the classical estimator of the two-parameter FD based on type-I censored data. We obtain the ML estimators and it is observed that the ML estimators cannot be obtained in explicit form. Based on the simulation results the following conclusions are drawn. For the parameters α and β, the MLEs $\acute{\alpha}$ and $\acute{\beta}$ have smaller variances as the sample size increases. The CIs for $\acute{\alpha}$ and $\acute{\beta}$ had a small interval of the estimates for both parametric combination (3, 1.5) produces good results as compare to other combination of parametric values. We have assumed four different terminating time i.e., t_c, and it is observed from the results that at higher terminating time $t_c = 10$, the estimates are more closer to their true values. From simulation results it is concluded that the ML estimates have good statistical properties.

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